

# LENGTH MAXIMIZING INVARIANT MEASURES IN LORENTZIAN GEOMETRY

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**ABSTRACT.** We introduce a version of Aubry-Mather theory for the length functional of causal curves in a compact Lorentzian manifold. Results include the existence of maximal invariant measures, calibrations and calibrated curves. We prove two versions of Mather's graph theorem for Lorentzian manifolds. A class of examples (Lorentzian Hedlund examples) shows the optimality of the results.

## 1. INTRODUCTION

Besides the theory of closed geodesics, Aubry-Mather theory provides an additional possibility for studying the geodesic flow of a general compact Riemannian manifold. The theory of closed geodesics in Lorentzian geometry is an active field of research with recent new and astonishing developments. In the present paper, however, we want to direct the attention to an Aubry-Mather theory for Lorentzian manifolds. This attempt is very natural in view of the geometric character of Aubry-Mather theory. The minimality assumptions on the curves in the positive definite case translate readily to a maximality assumption on causal curves in Lorentzian manifolds.

So far, there have been two previous attempts ([19], [21]) towards an Aubry-Mather theory in Lorentzian geometry. These notes will generalize both works towards a much larger class of Lorentzian manifolds. A short account of these previous results is contained in section 3.

The prototype for the theory we intend is developed in [11], [2], [6] and [3]. We will generalize the important results of these articles to the naturally given class of so-called class A spacetimes. A compact Lorentzian manifold  $(M, g)$  is of class A if it is (1) time orientable, i.e. it gives rise to a continuous timelike vector field, (2) it is vicious, i.e. every point lies on a timelike loop and (3) the Abelian cover is globally hyperbolic (see definition 2.2). In a rough sense this could be seen as a minimal catalogue of requirements a Lorentzian manifold has to satisfy in order to support an Aubry-Mather theory.

The results in these notes include an adequate analogue, called the stable time separation, of the stable norm (or Mather's  $\beta$ -function), the relations between the convexity properties of the stable time separation and maximal causal geodesics in the Abelian cover of  $(M, g)$ . Further we prove the existence of calibrations for class A spacetimes and the analogue of the Mather graph theorem. Finally we introduce the Lorentzian Hedlund examples (for the Riemannian case see [2]). These examples give an idea in what sense the obtained results are optimal.

The article is organized as follows. First we briefly review the necessary tools and results in Lorentzian geometry. In section 3 we give an account of previous Aubry-Mather theories in Lorentzian geometry.

In section 4 we introduce the stable time separation and prove its first properties. The rest of the section is devoted to finite Borel measures  $\mu$  on  $TM$  with support

contained in the set of future pointing tangent vectors and invariant under a suitable reparameterization of the geodesic flow of  $(M, g)$ . We define the *average length* of  $\mu$  (analogous to the action)

$$\mathfrak{L}(\mu) := \int_{T^1, R_M} \sqrt{|g(v, v)|} d\mu(v)$$

and the rotation class  $\rho(\mu) \in H_1(M, \mathbb{R})$  of  $\mu$  like in [11]. The existence of maximal invariant measures in a given homology class  $h$  in the stable time cone  $\mathfrak{T}$  follows from the properties of class A spacetimes (for the definition of  $\mathfrak{T}$  see [20] and section 2.1.1).

In section 5 we define calibrations in order to understand the relation between maximal invariant measures and maximizers of  $(M, g)$ , i.e. future pointing pregeodesics which lift to maximal pregeodesics in the Abelian cover. Here calibrations are Lipschitz time functions on the Abelian cover equivariant under the action of  $H_1(M, \mathbb{Z})_{\mathbb{R}}$  and growing with the least amount possible along future pointing curves. We verify the existence of calibrations for class A spacetimes. Our approach is similar to the one given in [6].

Section 6 then studies the relationships between maximal measures and future pointing maximizers of  $(M, g)$ . We prove that every pregeodesic whose tangent curve is contained in the support of a maximal measure with rotation class contained in  $\mathfrak{T}^\circ$  is a maximizer. Further we prove that any calibrated curve (for the definition see section 6) is timelike and the tangents are bounded away from the light cones in  $TM$ . The existence of calibrations and calibrated curves then proves that every class A spacetime contains at least one timelike maximizer such that the closure of its tangents is contained in the timelike vectors.

One of the grand results in Aubry-Mather theory is the so-called Lipschitz graph theorem in [11]. It states that the projection  $\pi: TM \rightarrow M$  restricted to the support of any minimal measure is injective and the inverse of the restriction is Lipschitz continuous. The proof relies on a shortening principle for minimizers. The idea is local in nature and obvious for self-intersecting curves. Surprisingly the estimate is true for minimal curves passing each other with a bound on the distance of the directions relative to the distance of the base points. This bound in turn is responsible for the Lipschitz continuity of  $(\pi|_{\text{supp } \mathfrak{M}_\alpha})^{-1}$ .

The picture changes for the problem of maximal measures in the Lorentzian case. For general maximal measures we prove a  $1/2$ -Hölder continuity of  $(\pi|_{\text{supp } \mathfrak{M}_\alpha})^{-1}$  in section 7. With the present techniques this result is optimal in the general case. Whether there exists a non-local argument and if this can be applied is the subject of further research. Contrary if we bound the support of the maximal measures away from the light cones, we retain the Lipschitz continuity of  $(\pi|_{\text{supp } \mathfrak{M}_\alpha})^{-1}$ . The difference to the general case is mostly due to local connectivity arguments true with less strict assumptions in the timelike case than in the general case.

Finally in section 8 we introduce a family of class A spacetimes structures on  $T^3$  called the Lorentzian Hedlund examples. The construction shows that the results on the multiplicity of maximizing invariant measures are optimal. More precisely the Hedlund examples contain exactly three maximal invariant measures supported on three closed timelike geodesics. Additionally we provide a precise analysis of the behavior of the timelike maximizers relevant for the Aubry-Mather theory of these examples. The Riemannian counterparts of the Lorentzian Hedlund examples are discussed in [2].

## 2. PRELIMINARIES

Throughout the entire text we will assume that a complete Riemannian metric  $g_R$  on  $M$  has been chosen. We denote the distance function relative to  $g_R$  by  $\text{dist}$

and the metric balls of radius  $r$  around  $p \in M$  with  $B_r(p)$ . The metric  $g_R$  induces a norm on every tangent space of  $M$  which we denote by  $|\cdot|$ . For convenience of notation we denote the lift of  $g_R$  to  $\overline{M}$ , and all objects associated to it, with the same letter. Denote with  $\text{diam}(M, g_R)$  the diameter of a fundamental domain of the Abelian cover.

**2.1. Lorentzian Geometry.** The following concepts are basic notions in Lorentzian geometry. For details we refer to the standard textbook references [8], [16] and [4]. For the recent developments in causality theory see [14].

We consider only connected manifolds. Recall that a compact manifold  $M$  admits a Lorentzian metric if and only if  $\chi(M) = 0$ . In comparison every noncompact manifolds admits Lorentzian metrics.

Define the space

$$\text{Lor}(M) := \{\text{smooth Lorentzian metrics on } M\} \subseteq \Gamma^\infty(T_2^0 M).$$

A topology on  $\text{Lor}(M)$  is induced by the fine  $C^0$ -topology on continuous sections of  $T_2^0 M$  (see [8], p. 198).

**Definition 2.1** ([4]). *A Lorentzian manifold  $(M, g)$  is a spacetime if it is time-oriented.*

Note that every Lorentzian manifold admits a twofold time orientable cover.

**Definition 2.2** ([4]). *Let  $(M, g)$  be a spacetime.*

- (1)  *$(M, g)$  is causal if  $p \notin J^+(p)$  for all  $p \in M$ .*
- (2)  *$(M, g)$  is globally hyperbolic if  $(M, g)$  is causal and the intersections  $J^+(p) \cap J^-(q)$  are compact for all  $p, q \in M$ .*
- (3)  *$(M, g)$  is vicious at  $p \in M$  if  $M = I^+(p) \cap I^-(p)$ .*

Note that viciousness does not depend on the particular point  $p \in M$ . For example see [4], lemma 4.2.:  $(M, g)$  is vicious at every point in  $M$  if and only if  $(M, g)$  is vicious at one point. Further note that the given definition of viciousness is obviously equivalent to the condition that every point lies on a timelike loop.

**Definition 2.3** ([14]). *Let  $(M, g)$  be a spacetime. A function  $\tau: M \rightarrow \mathbb{R}$  is a*

- (i) *time function if  $\tau$  is continuous and strictly increasing along every future pointing curve in  $(M, g)$ .*
- (ii) *temporal function if  $\tau$  is a smooth function with past pointing timelike gradient  $\nabla\tau$ .*

**Definition 2.4** ([4]). *Let  $(M, g)$  be a Lorentzian manifold and  $\gamma: [a, b] \rightarrow M$  a causal curve. Then we define the length of  $\gamma$ :*

$$L^g(\gamma) := \int_a^b \sqrt{|g(\dot{\gamma}(t), \dot{\gamma}(t))|} dt$$

We have the following upper semicontinuity for the Lorentzian length functional.

**Proposition 2.5** ([4]). *If a sequence of causal curves  $\gamma_n: [a, b] \rightarrow M$ , parameterized w.r.t.  $g_R$ -arclength, converges uniformly to the causal curve  $\gamma: [a, b] \rightarrow M$ , then*

$$L^g(\gamma) \geq \limsup L^g(\gamma_n).$$

**Definition 2.6** ([4]). *The time separation or Lorentzian distance function is defined as  $d(p, q) := \sup\{L^g(\gamma) \mid \gamma \in \mathcal{C}^+(p, q)\}$  with the convention  $\sup \emptyset := 0$ .*

Naturally a future pointing curve  $\gamma: I \rightarrow M$  is said to be maximal if

$$L^g(\gamma|_{[s, t]}) = d(\gamma(s), \gamma(t))$$

for all  $s \leq t \in I$ .

**Corollary 2.7** ([4]). *Let  $(M, g)$  be globally hyperbolic. Then the time separation is continuous and there exists a maximal causal geodesic connecting  $p$  with  $q$  for all  $q \in J^+(p)$ .*

Denote by  $[g]$  the conformal class of the Lorentzian metric  $g$  sharing the same time-orientation. Define the sets

$$\text{Time}(M, [g]) := \{\text{future pointing timelike vectors in } (M, g)\}$$

and

$$\text{Light}(M, [g]) := \{\text{future pointing lightlike vectors in } (M, g)\}.$$

Both  $\text{Time}(M, [g])$  and  $\text{Light}(M, [g])$  are smooth fibre bundles over  $M$ . Denote by  $\text{Time}(M, [g])_p$  and  $\text{Light}(M, [g])_p$  the fibres of  $\text{Time}(M, [g])$  and  $\text{Light}(M, [g])$  over  $p \in M$ , respectively. For  $\varepsilon > 0$  set

$$\text{Time}(M, [g])^\varepsilon := \{v \in \text{Time}(M, [g]) \mid \text{dist}(v, \text{Light}(M, [g])) \geq \varepsilon|v|\}.$$

$\text{Time}(M, [g])^\varepsilon$  is a smooth fibre bundle as well with fibre  $\text{Time}(M, [g])_p^\varepsilon$  over  $p \in M$ . Note that the fibres are convex for every  $p \in M$ .

**2.1.1. Causality Properties of Class A Spacetimes.** The results of this section are the subject of [20]. For details we refer to [20].

**Definition 2.8.** *A compact spacetime  $(M, g)$  is of class A if  $(M, g)$  is vicious and the Abelian cover  $\bar{\pi}: (\bar{M}, \bar{g}) \rightarrow (M, g)$  is globally hyperbolic.*

Before we can recall the definition of the *stable time cone* we need the concept of *rotation vectors* ([11]). Let  $k_1, \dots, k_b$  ( $b := \dim H_1(M, \mathbb{R})$ ) be a basis of  $H_1(M, \mathbb{R})$  consisting of integer classes, and  $\alpha_1, \dots, \alpha_b$  the dual basis with representatives  $\omega_1, \dots, \omega_b$ . For two points  $x, y \in \bar{M} := \widetilde{M}/[\pi_1(M), \pi_1(M)]$  we define the *difference*  $y - x \in H_1(M, \mathbb{R})$  via a  $C^1$ -curve  $\gamma: [a, b] \rightarrow \bar{M}$  connecting  $x$  and  $y$ , by

$$\langle \alpha_i, y - x \rangle := \int_\gamma \pi^* \omega_i$$

for all  $i \in \{1, \dots, b\}$ . We define the rotation vector of  $\gamma$  as well as of  $\bar{\pi} \circ \gamma$ :

$$\rho(\gamma) := \frac{1}{b-a}(y-x).$$

Note that the map  $(x, y) \mapsto y - x$  is i.g. not surjective. But we know that the convex hull of the image is all of  $H_1(M, \mathbb{R})$ . Just observe that by our choice of classes  $\alpha_i$  we know that every  $k \in H_1(M, \mathbb{Z})_{\mathbb{R}}$  (image of  $H_1(M, \mathbb{Z}) \rightarrow H_1(M, \mathbb{R})$ ) is the image of  $(x, x + k)$  for every  $x \in \bar{M}$ .

Now a sequence of causal curves  $\{\gamma_i\}_{i \in \mathbb{N}}$  is called *admissible*, if  $L^{g_R}(\gamma_i) \rightarrow \infty$  for  $i \rightarrow \infty$ .  $\mathfrak{T}^1$  is defined to be the set of all accumulation points of sequences  $\{\rho(\gamma_i)\}_{i \in \mathbb{N}}$  in  $H_1(M, \mathbb{R})$  of admissible sequences  $\{\gamma_i\}_{i \in \mathbb{N}}$ .  $\mathfrak{T}^1$  is compact for any compact spacetime since the stable norm of any rotation vector is bounded by  $1 + \text{std}(g_R)$ . Note that if  $(M, g)$  is vicious,  $\mathfrak{T}^1$  is convex by the following fact.

**Fact 2.9.** *Let  $M$  be compact and  $(M, g)$  vicious. Then there exists a constant  $\text{fill}(g, g_R) < \infty$  such that any two points  $p, q \in M$  can be joined by a future pointing timelike curve with  $g_R$ -arclength less than  $\text{fill}(g, g_R)$ .*

We define the *stable time cone*  $\mathfrak{T}$  to be the cone over  $\mathfrak{T}^1$ . Note that  $\mathfrak{T}$  does not depend on the choice of  $g_R$ ,  $\{k_1, \dots, k_b\}$  and  $\omega_i \in \alpha_i$ , whereas  $\mathfrak{T}^1$  does. Reversing the time-orientation yields  $-\mathfrak{T}$  as stable time cone.  $\mathfrak{T}$  is invariant under global conformal changes of the metric and therefore depends only on the causal structure of  $(M, g)$ , i.e. the distribution of lightcones. Note as well that for compact and

vicious spacetimes  $\mathfrak{T}$  is equal to the closure of the cone over the homology classes of future pointing causal loops.

For compact and vicious spacetimes the stable time cone is characterized uniquely by the following property.

**Proposition 2.10.** *Let  $(M, g)$  be a compact and vicious spacetime. Then  $\mathfrak{T}$  is the unique cone in  $H_1(M, \mathbb{R})$  such that there exists a constant  $\text{err}(g, g_R) < \infty$  with  $\text{dist}_{\|\cdot\|}(J^+(x) - x, \mathfrak{T}) \leq \text{err}(g, g_R)$  for all  $x \in \overline{M}$ , where  $J^+(x) - x := \{y - x \mid y \in J^+(x)\}$ .*

By  $\mathfrak{T}^*$  we denote the *dual stable time cone*, i.e.

$$\mathfrak{T}^* := \{\alpha \in H^1(M, \mathbb{R}) \mid \alpha|_{\mathfrak{T}} \geq 0\}.$$

The following theorem is the first main result of [20].

**Theorem 2.11.** *Let  $(M, g)$  be compact and vicious. Then the following statements are equivalent:*

- (i)  $(M, g)$  is of class A.
- (ii)  $0 \notin \mathfrak{T}^1$ , especially  $\mathfrak{T}$  contains no linear subspaces.
- (iii)  $(\mathfrak{T}^*)^\circ \neq \emptyset$  and for every  $\alpha \in (\mathfrak{T}^*)^\circ$  there exists a smooth 1-form  $\omega$  representing  $\alpha$  such that  $\ker \omega_p$  is a spacelike hyperplane in  $(TM_p, g_p)$  for all  $p \in M$ .

Notable corollaries of this theorem are the above mentioned openness of the set of class A metrics in  $\text{Lor}(M)$  relative to the uniform topology and the topological characterization of class A spacetimes as mapping tori.

**Corollary 2.12.** *Let  $(M, g)$  be of class A. Then there exists a constant  $C_{g, g_R} < \infty$  such that*

$$L^{\overline{g}_R}(\gamma) \leq C_{g, g_R} \text{dist}(p, q)$$

for all  $p, q \in \overline{M}$  and  $\gamma$  a causal curve connecting  $p$  with  $q$ .

For  $p \in M$  let  $\mathfrak{T}_p$  be the set of classes  $k \in H_1(M, \mathbb{Z})_{\mathbb{R}}$  which can be represented by a timelike future pointing loop through  $p$ . A homology class  $h \in H_1(M, \mathbb{R})$  is called  $\mathfrak{T}_p$ -rational if  $nh \in \mathfrak{T}_p$  for some positive integer  $n$ .

**Proposition 2.13.** *For every  $R > 0$  there exists a constant  $K = K(R) < \infty$  such that*

$$B_R(q) \subseteq I^+(p)$$

for all  $p, q \in \overline{M}$  with  $q - p \in \mathfrak{T}$  and  $\text{dist}_{\|\cdot\|}(q - p, \partial\mathfrak{T}) \geq K$ .

The second main result of [20] concerns the coarse-Lipschitz property of the time separation of the Abelian cover of a class A spacetime. For  $\varepsilon > 0$  set  $\mathfrak{T}_\varepsilon := \{h \in \mathfrak{T} \mid \text{dist}_{\|\cdot\|}(h, \partial\mathfrak{T}) \geq \varepsilon \|h\|\}$ .

**Theorem 2.14.** *Let  $(M, g)$  be of class A. Then for every  $\varepsilon > 0$  there exists  $L_c(\varepsilon) < \infty$ , such that*

$$|d(x, y) - d(z, w)| \leq L_c(\varepsilon)(\text{dist}(x, z) + \text{dist}(y, w) + 1)$$

for all  $(x, y), (z, w) \in \overline{M} \times \overline{M}$  with  $y - x, w - z \in \mathfrak{T}_\varepsilon$ .

### 3. PREVIOUS WORK

There are two predecessors of the theory developed in this article. The diploma thesis [19] studies the case of Lorentzian surfaces, whereas [21] is concerned with globally conformally flat Lorentzian tori of arbitrary dimension. The methods used in [19] are translations of methods used in [1]. The methods applied in [21] are taken from [2] and [5].

**3.1. Lorentzian 2-tori.** In this section we consider closed orientable surfaces  $M^2$  with vanishing Euler characteristic, i.e.  $M^2 \cong T^2$ . For a discussion of Lorentzian metrics on closed surfaces see [22].

Assume that the (locally well defined) lightlike distributions are well defined and orientable, i.e. there exist two future pointing lightlike vector fields  $X^+$  and  $X^-$  such that  $\{X_p^+, X_p^-\}$  is a positive oriented basis of  $TM_p^2$  for all  $p \in M^2$  (this is true up to a twofold covering, see [22]).

Recall the definition of  $m^\pm$  from [22]. For every integral curve  $\zeta^\pm: \mathbb{R} \rightarrow M$  of  $X^\pm$  set  $m^\pm := \lim_{T \rightarrow \infty} [\text{span}(\zeta^\pm(T) - \zeta^\pm(0))] \in PH_1(M, \mathbb{R})$ .

We are only interested in the case  $m^+ \neq m^-$ . Under these assumptions, if the lightlike curve  $\zeta$  is future pointing, all homology classes  $\zeta(T_2) - \zeta(T_1)$  ( $T_1 \leq T_2$ ) lie in a bounded distance of a halfline  $\overline{m}^\pm$  of  $m^\pm$ . Consequently, instead of the projective class  $m^\pm$ , only a halfline of  $m^\pm$  needs to be considered to distinguish the asymptotic direction of the lightlike distributions. Denote by  $\mathfrak{T}$  the convex hull of  $\overline{m}^+ \cup \overline{m}^-$ . This definition of  $\mathfrak{T}$  coincides with the general definition in the previous section.

According to [22], the condition  $m^+ \neq m^-$  is equivalent to  $(M, g)$  being class A. Note that the condition  $m^+ \neq m^-$  has no obvious counterpart in dimensions  $\geq 3$ . This is due to the fact that only in dimension 2 the light cones are given as the union of two linear subspaces of the tangent space. For this reason we follow a different approach to the causality conditions for class A spacetimes.

**Lemma 3.1** ([19] Lemma 4.3). *Let  $h \in \mathfrak{T} \cap H_1(M^2, \mathbb{Z})$ . Then there exists a closed maximizer  $\gamma: \mathbb{R} \rightarrow M^2$  with homology class  $h$ .*

The proof is an adaptation of [10]. It includes a maximization argument on the space of closed causal curves representing  $h$ . As in [10] the proof gives further information on the minimal period of the maximizers:

**Corollary 3.2** ([19] Korollar 4.4). *Let  $\gamma: \mathbb{R} \rightarrow M^2$  be a closed timelike maximizer with minimal period  $T > 0$ . Set  $h := [\gamma|_{[0, T]}] \in H_1(M^2, \mathbb{Z})$ . Then the class  $h$  is relative prim in  $H_1(M^2, \mathbb{Z})$ , i.e. for any  $h' \in H_1(M^2, \mathbb{Z})$  and  $\lambda > 0$  with  $h = \lambda h'$  we have  $\lambda = 1$  and  $h' = h$ .*

**Theorem 3.3** ([19] Satz 4.2). *For any one dimensional half space  $\overline{m} \subseteq \mathfrak{T}$  there exists a timelike maximizer  $\gamma: \mathbb{R} \rightarrow M^2$  such that for all  $T_1 \leq T_2$  the homology class  $\gamma(T_2) - \gamma(T_1)$  lies at bounded distance from  $\overline{m}$ . This distance depends only on  $(M^2, g)$ .*

**Main Result** ([19]). (1) *For every causal maximizer there exists a half space  $m \subseteq \mathfrak{T}$  such that all rotation vectors of all intervals lie at a bounded distance from  $m$ . Call  $m$  the asymptotic direction of the maximizer.*

(2) *Given a pair of geometrically distinct maximizers  $(\gamma_1, \gamma_2)$  such that the asymptotic directions of  $\gamma_1$  and  $\gamma_2$  coincide. If the asymptotic direction is  $H_1(M^2, \mathbb{Z})$ -irrational then  $\gamma_1$  and  $\gamma_2$  do not intersect, i.e. are disjoint.*

(3) *Given an irrational halfline  $m \subseteq H_1(M^2, \mathbb{R})$  consider the set of points in  $M^2$  lying on a maximizer with asymptotic direction  $m$ . Then this set is either all of  $M^2$  or it intersects every transversal in a Cantor set.*

(4) *In every strip between neighboring periodic maximizers  $\gamma_1, \gamma_2$  there exist maximizers either  $\alpha$ -asymptotic to  $\gamma_1$  and  $\omega$ -asymptotic to  $\gamma_2$  or  $\alpha$ -asymptotic to  $\gamma_2$  and  $\omega$ -asymptotic to  $\gamma_1$ .*

(5) *Every non-periodic maximizer with rational asymptotic direction is of one of these types.*

**3.2. Lorentzian conformally-flat  $n$ -tori.** The reference for these results is [21]. Consider a real vector space  $V$  of dimension  $m < \infty$  and  $\langle \cdot, \cdot \rangle_1$  a nondegenerate



symmetric bilinear form on  $V$  with signature  $(-, +, \dots, +)$ . Further let  $\Gamma \subseteq V$  be a co-compact lattice and  $f: V \rightarrow (0, \infty)$  a smooth and  $\Gamma$ -invariant function. The Lorentzian metric  $\bar{g} := f^2 \langle \cdot, \cdot \rangle_1$  then descends to a Lorentzian metric on the torus  $V/\Gamma$ . Denote the induced Lorentzian metric by  $g$ . Choose a time orientation of  $(V, \langle \cdot, \cdot \rangle_1)$ . This time orientation induces a time orientation on  $(V/\Gamma, g)$  as well. Note that  $(V/\Gamma, g)$  is vicious and the universal cover  $(V, \bar{g})$  is globally hyperbolic. According to [17] proposition 2.1,  $(V/\Gamma, g)$  is geodesically complete in all three causal senses. Fix a norm  $\|\cdot\|$  on  $V$  and denote the dual norm by  $\|\cdot\|^*$ . Note that a norm on  $V$  is equal to its stable norm on  $H_1(V/\Gamma, \mathbb{R})$  via the canonical identification  $V \cong H_1(V/\Gamma, \mathbb{R})$ . Note that  $\|\cdot\|$  induces a metric on  $V/\Gamma$ . For a subset  $A \subseteq V$  we write  $\text{dist}(x, A)$  to denote the distance of the point  $x \in V$  to  $A$  relative to  $\|\cdot\|$ . Then  $\mathfrak{T}$  is equal to the set of futurepointing causal vectors in  $(V, \langle \cdot, \cdot \rangle_1)$  and we have  $J^+(p) = p + \mathfrak{T}$  for all  $p \in V$ .

Choose an orthonormal basis  $\{e_1, \dots, e_m\}$  of  $(V, \langle \cdot, \cdot \rangle_1)$ . Note that the translations  $x \mapsto x + v$  are conformal diffeomorphisms of  $(V, \bar{g})$  for all  $v \in V$ . Then the  $\bar{g}$ -orthogonal frame field  $x \mapsto (x, (e_1, \dots, e_m))$  on  $V$  descends to a  $g$ -orthogonal frame field on  $V/\Gamma$ . Relative to this identification of  $V \cong TV_p$  we get  $\mathfrak{T} = \text{Time}(V, [\bar{g}])_p \cup \text{Light}(V, [\bar{g}])_p$  and  $\mathfrak{T}_\varepsilon = \text{Time}(V, [\bar{g}])_p^\varepsilon$ .

For a continuous curve  $\gamma: I \rightarrow V/\Gamma$  and  $s, t \in I$  set  $\gamma(t) - \gamma(s) := \bar{\gamma}(t) - \bar{\gamma}(s)$ , where  $\bar{\gamma}: I \rightarrow V$  is any lift of  $\gamma$ . Note that this definition of *difference* coincides with the general definition we gave 2.1.1.

Obviously the causal structure of these spacetimes is much simpler in comparison to class A spacetimes. This eliminates all problems one faces with causality considerations in more general spacetimes.

**Proposition 3.4.** *There exists a positively homogenous concave function  $\mathfrak{l}: \mathfrak{T} \rightarrow [0, \infty)$  such that:*

- (1) *For every  $\varepsilon > 0$  there exists  $K(\varepsilon) < \infty$  such that*

$$|\mathfrak{l}(v) - d(x, x + v)| \leq K(\varepsilon)$$

*for all  $v \in \mathfrak{T}_\varepsilon$  and all  $x \in V$ .*

- (2)  *$\inf f \sqrt{|\langle v, v \rangle_1|} \leq \mathfrak{l}(v) \leq \sup f \sqrt{|\langle v, v \rangle_1|}$  for all  $v \in \mathfrak{T}$ .*

- (3)  *$\mathfrak{l}(v + w) \geq \mathfrak{l}(v) + \mathfrak{l}(w)$  for all  $v, w \in \mathfrak{T}$ .*

Define the rotation vector of a future pointing curve  $\gamma: [a, b] \rightarrow V/\Gamma$ :

$$R(\gamma) := \frac{1}{\mathfrak{l}(\gamma(b) - \gamma(a))} [\gamma(b) - \gamma(a)]$$

**Theorem 3.5.** *Let  $\varepsilon > 0$  and  $\gamma: \mathbb{R} \rightarrow V/\Gamma$  be a maximizer with  $\dot{\gamma}(t_0) \in \mathfrak{T}_\varepsilon$  for some  $t_0 \in \mathbb{R}$ . Then there exists a support function  $\alpha$  of  $\mathfrak{l}$  such that for all neighborhoods  $U$  of  $\alpha^{-1}(1) \cap \mathfrak{l}^{-1}(1)$  there exists  $K = K(\varepsilon, U) < \infty$  such that for all  $s < t \in \mathbb{R}$  with  $\|\gamma(t) - \gamma(s)\| \geq K$ , we have*

$$R(\gamma|_{[s, t]}) \in U.$$

Note that the function  $\tau(p) := \alpha(p)$  is a temporal function on  $(V, \bar{g})$  iff  $\alpha \in (\mathfrak{T}^*)^\circ$ . We call a function  $\tau: \bar{M} \rightarrow \mathbb{R}$   $\alpha$ -equivariant if  $\tau$  is equivariant under the action of  $\Gamma$  on  $V$  and of  $\alpha(\Gamma)$  on  $\mathbb{R}$ , i.e.  $\tau(x + k) = \tau(x) + \alpha(k)$  for all  $k \in \Gamma$  and all  $x \in V$ .

**Definition 3.6.** *Let  $\alpha \in (\mathfrak{T}^*)^\circ$  and  $\tau: V \rightarrow \mathbb{R}$  be an  $\alpha$ -equivariant temporal function.*

- (1) *Define for  $\sigma \in \mathbb{R}$ :*

$$\mathfrak{h}_\tau(\sigma) := \sup \{L^g(\gamma) \mid \gamma \text{ future pointing, } \int_\gamma \omega_\tau = \sigma\}$$

- (2) A future pointing maximizer  $\gamma: I \rightarrow V/\Gamma$  is said to be  $\alpha$ -almost maximal if there exists a constant  $F < \infty$  such that

$$L^g(\gamma|_{[s,t]}) \geq \mathfrak{h}_\tau \left( \int_{\gamma|_{[s,t]}} \omega_\tau \right) - F$$

for all  $s < t \in I$ .

Denote by  $\mathfrak{l}^*: \mathfrak{T}^* \rightarrow \mathbb{R}$  the dual function of  $\mathfrak{l}$ , i.e.  $\mathfrak{l}(\alpha) := \min\{\alpha(v) \mid \mathfrak{l}(v) = 1\}$ .

- Theorem 3.7.** (1) For every  $\alpha \in (\mathfrak{T}^*)^\circ$  there exists an  $\alpha$ -almost maximal timelike geodesic  $\gamma: \mathbb{R} \rightarrow V/\Gamma$ .  
 (2) Let  $\alpha \in \mathfrak{T}^*$  with  $\mathfrak{l}^*(\alpha) = 1$ . Then for every neighborhood  $U$  of  $\alpha^{-1}(1) \cap \mathfrak{l}^{-1}(1)$  there exists  $K = K(\alpha, U) < \infty$  such that

$$R(\gamma|_{[s,t]}) \in U$$

for all  $\alpha$ -almost maximal future pointing curves  $\gamma: \mathbb{R} \rightarrow V/\Gamma$  and every  $s < t \in \mathbb{R}$  with  $\|\gamma(t) - \gamma(s)\| \geq K$ .

**Corollary 3.8.**  $(V/\Gamma, g)$  contains infinitely many geometrically distinct timelike maximizers  $\gamma: \mathbb{R} \rightarrow V/\Gamma$  with the additional property that the limit

$$\lim_{t \rightarrow \infty} R(\gamma|_{[s, s+t]}) =: v$$

exists uniformly in  $s \in \mathbb{R}$  and that these limits  $v$  are exposed points of  $\mathfrak{l}^{-1}(1)$ .

Corollary 3.8 shows the main difference to the results of [2]. [19] contains a similar result for Lorentzian surfaces. But there the fact that the spacetimes are of dimension 2 plays the crucial role. Corollary 3.8 opposes the results obtained in connection with the Hedlund example for the Riemannian case. Note that the Riemannian Hedlund examples already exist in the conformal class of the flat metric. 3.8 shows that this is not true for globally conformally flat Lorentzian metrics. But to achieve the necessary phenomenon in the Lorentzian case, one has to distort the causal structure as well.

#### 4. THE STABLE TIME SEPARATION

We have the following analogue of the stable norm for class A spacetimes.

**Theorem 4.1.** Let  $(M, g)$  be of class A. Then there exists a unique concave function  $\mathfrak{l}: \mathfrak{T} \rightarrow \mathbb{R}$  such that for every  $\varepsilon > 0$  there is a constant  $\overline{C}(\varepsilon) < \infty$  with

- (1)  $|\mathfrak{l}(h) - d(x, y)| \leq \overline{C}(\varepsilon)$  for all  $x, y \in \overline{M}$  with  $y - x = h \in \mathfrak{T}_\varepsilon$  and
- (2)  $\mathfrak{l}(\lambda h) = \lambda \mathfrak{l}(h)$ , for all  $\lambda \geq 0$ ,
- (3)  $\mathfrak{l}(h' + h) \geq \mathfrak{l}(h') + \mathfrak{l}(h)$  for all  $h, h' \in \mathfrak{T}$  and
- (4)  $\mathfrak{l}(h) = \limsup_{h' \rightarrow h} \mathfrak{l}(h')$  for  $h \in \partial \mathfrak{T}$  and  $h' \in \mathfrak{T}$ .

We will call  $\mathfrak{l}$  the stable time separation.

**Remark 4.2.** Property (4) in theorem 4.1 will become apparent in the next section with the treatment of invariant measures.

For the proof we will follow the steps in [5].

**Definition 4.3.** Let  $(M, g)$  be of class A. For  $x \in \overline{M}$  and  $h \in H_1(M, \mathbb{Z})_\mathbb{R}$  set

$$\mathfrak{d}(h) := \sup\{d(x, x+h) \mid x \in \overline{M}\}.$$

Since the time separation  $d$  is invariant under the group of deck transformation the function  $\mathfrak{d}(h)$  is well defined.

**Lemma 4.4.** Let  $(M, g)$  be of class A. Then for all  $\varepsilon > 0$  there exists a  $D(\varepsilon) < \infty$ , such that



- (1)  $z\mathfrak{d}(h) \leq \mathfrak{d}(zh)$  for  $z = 2, 3$  and
- (2)  $2\mathfrak{d}(h) \geq \mathfrak{d}(2h) - D(\varepsilon)$

for all  $h \in \mathfrak{T}_\varepsilon \cap H_1(M, \mathbb{Z})_\mathbb{R}$ .

*Proof.* No new ideas are necessary. Theorem 2.14 and fact 2.9 are sufficient to follow the steps in [5] to prove the properties.  $\square$

The following lemma is the analogous version of lemma 1 in [5].

**Lemma 4.5.** *Let  $C < \infty$  and  $F: \mathbb{N} \rightarrow [0, \infty)$  be a coarse-Lipschitz function with*

- (1)  $2F(s) - F(2s) \geq -C$ ,
- (2)  $F(\kappa s) - \kappa F(s) \geq -C$ , for  $\kappa = 2, 3$

*and all  $s \in \mathbb{N}$ . Then there exists  $\mathfrak{d} \in \mathbb{R}$  such that  $|F(s) - \mathfrak{d}s| \leq 2C$  for all  $s \in \mathbb{N}$ .*

*Proof of theorem 4.1.* (1) follows directly with lemma 4.4 and 4.5 for  $h \in \mathfrak{T}_\varepsilon \cap H_1(M, \mathbb{Z})_\mathbb{R}$  and  $y = x + h$ . The general case then follows with theorem 2.14, fact 2.9 and the usual cut-and-paste arguments.

The proof of (2) and (3) for  $h, h' \in \mathfrak{T}^\circ$  follow in the same fashion as shown in [2]. If we define  $\mathfrak{l}|_{\partial\mathfrak{T}}$  by property (4), (2) and (3) follow directly for  $\mathfrak{l}|_{\partial\mathfrak{T}}$ .  $\square$

We call a future pointing curve  $\gamma: [a, b] \rightarrow \overline{M}$  a maximizer if  $\gamma$  maximizes ar-length over all future pointing curves connecting  $\gamma(a)$  with  $\gamma(b)$ . For the convenience of notation we call  $\gamma: [a, b] \rightarrow M$  a maximizer if one (hence every) lift to  $\overline{M}$  is a maximizer. A future pointing curve  $\gamma: \mathbb{R} \rightarrow M$  (or  $\overline{M}$ ) is a maximizer if the restriction  $\gamma|_{[a, b]}$  is a maximizer for every finite interval  $[a, b] \subseteq \mathbb{R}$ .

**Remark 4.6.** *For any  $h \in \mathfrak{T}$  there exists a sequence of maximizers  $\{\gamma_n\}_{n \in \mathbb{N}}$  and a  $\lambda > 0$  such that  $\lambda(\rho(\gamma_n), L^g(\gamma_n)) \rightarrow (h, \mathfrak{l}(h))$ .*

**Corollary 4.7.** *Consider an admissible sequence  $\gamma_n: [a_n, b_n] \rightarrow M$  ( $n \in \mathbb{N}$ ) of maximizers and suppose that  $\rho(\gamma_n) \rightarrow h \in \mathfrak{T}^\circ$ . Then we have*

$$\frac{L^g(\gamma_n)}{b_n - a_n} \rightarrow \mathfrak{l}(h),$$

for  $n \rightarrow \infty$ .

**Remark 4.8.** *The corollary extends to  $\mathfrak{T}$  if  $\mathfrak{l}|_{\partial\mathfrak{T}} \equiv 0$ . However if  $\mathfrak{l}|_{\partial\mathfrak{T} \setminus \{0\}} > 0$  we can easily construct a counterexample from the Hedlund examples in section 8.*

**Proposition 4.9.** *Let  $(M, g)$  be of class A. Assume that there exists  $\alpha \in \partial\mathfrak{T}^*$  such that  $\alpha^{-1}(0) \cap \mathfrak{T} \cap H_1(M, \mathbb{Z})_\mathbb{R} = \emptyset$ . Then we have  $\mathfrak{l}|_{\alpha^{-1}(0) \cap \mathfrak{T}} \equiv 0$ .*

**Remark 4.10.** *Note that the assumptions apply especially to totally irrational  $\alpha \in \partial\mathfrak{T}^*$ .*

It will be convenient to employ the following theorem.

**Theorem 4.11** ([5]). *Let  $(M, g_R)$  be a compact Riemannian manifold. Then there exists a constant  $\text{std}(g_R) < \infty$  such that*

$$|\text{dist}(x, y) - \|y - x\|| \leq \text{std}(g_R)$$

for any pair  $x, y \in \overline{M}$ .

Denote with  $\text{inj}(M, g)_p$  the injectivity radius of  $(M, g)$  at  $p$  relative to  $g_R$  and  $\text{inj}(M, g) := \inf_{p \in M} \text{inj}(M, g)_p$ .

*Proof.* Consider  $\alpha \in \partial\mathfrak{T}^*$  such that  $\alpha^{-1}(0) \cap \mathfrak{T} \cap H_1(M, \mathbb{Z})_\mathbb{R} = \emptyset$ . Assume that there exists a homology class  $h \in \alpha^{-1}(0) \cap \mathfrak{T}$  with  $\mathfrak{l}(h) > 0$ .

Choose an admissible sequence  $\gamma_n: [a_n, b_n] \rightarrow M$  of maximal future pointing pregeodesics with  $|\dot{\gamma}_n| \equiv 1$  and

$$\frac{1}{b_n - a_n}(\rho(\gamma_n), L^g(\gamma_n)) \rightarrow (h, \mathfrak{l}(h)).$$

Since  $\mathfrak{l}(h) > 0$  there exists  $v \in \text{Time}(M, [g])$  and  $\varepsilon, \delta > 0$  such that

$$\frac{1}{b_n - a_n}(\dot{\gamma}_n)_\#(\mathcal{L}^1|_{[a_n, b_n]})(B_\varepsilon(v)) \geq \delta$$

for infinitely many  $n$ . Denote  $p := \pi(v)$  and choose a geodesically convex neighborhood  $U \subseteq M$  of  $p$  and a  $t \in (0, \text{inj}(M, g))$ . By diminishing  $\varepsilon$  and  $\delta$  we can assume that  $B_\varepsilon(v) \subseteq \text{Time}(M, [g])$  and  $B_\varepsilon(p) \subseteq I_U^+(\gamma_w(-t)) \cap I_U^-(\gamma_w(t))$  for every  $w \in B_\varepsilon(v)$ .

Consider the sets  $A_n := \{t \in [a_n, b_n] \mid \dot{\gamma}_n(t) \in B_\varepsilon(v)\}$  and their connected components  $\{A_{n,\nu}\}_{1 \leq \nu \leq r(n)}$ . Choose for every  $1 \leq \nu \leq r(n)$  one  $t_{n,\nu} \in A_{n,\nu}$ . Then the double sequence  $\gamma_n(t_{n,\nu+1}) - \gamma_n(t_{n,\nu})$  is bounded away from 0  $\in H_1(M, \mathbb{R})$ , because otherwise we could construct a nullhomologous timelike loop in  $(M, g)$ . The Lebesgue measure of an individual  $A_{n,\nu}$  is bounded from above by  $2\varepsilon$ . Therefore the number of connected components of  $A_n$  is bounded from below by  $\frac{\delta(b_n - a_n)}{2\varepsilon}$ . Now the number of connected components  $A_{n,\nu'}$  such that  $\text{dist}(A_{n,\nu'}, A_{n,\nu'+1}) > \frac{4\varepsilon}{\delta}$  is bounded from above by  $\frac{\delta(b_n - a_n)}{4\varepsilon}$ . Thus the number of connected components  $A_{n,\nu'}$  such that

$$\|\gamma_n(t_{n,\nu'+1}) - \gamma_n(t_{n,\nu'})\| \leq \frac{4\varepsilon}{\delta} + \text{std}(\text{gR})$$

is bounded from below by  $\frac{\delta(b_n - a_n)}{4\varepsilon}$ .

By the condition on  $\varepsilon$  we can deform  $\gamma_n|_{[a_n, t_{n,2}]}$  to a future pointing curve  $\gamma_n^1: [a_n, t_{n,2}] \rightarrow M$  homotopic with fixed endpoints to  $\gamma_n|_{[a_n, t_{n,2}]}$  and  $\gamma_n^1(t_{n,1}) = p$ . Continue this operation inductively for all  $1 \leq \nu \leq r(n)$ . This yields a future pointing curve  $\gamma_n^{r(n)}: [a_n, b_n] \rightarrow M$  homotopic with fixed endpoints to  $\gamma_n$  and  $\gamma_n^{r(n)}(t_{n,\nu}) = p$  for all  $1 \leq \nu \leq r(n)$ . Consequently we have

$$k_{n,\nu} := [\gamma_n^{r(n)}|_{[t_{n,\nu}, t_{n,\nu+1}]}] \in \mathfrak{T} \cap H_1(M, \mathbb{Z})_{\mathbb{R}}$$

and  $\alpha(k_{n,\nu}) \geq 0$  for all  $n$  and  $\nu$ , since  $\alpha$  is a support function of  $\mathfrak{T}$ . But then, since  $\alpha(\rho(\gamma_n)) \rightarrow 0$ , there exists a bounded sequence of  $\{k_{n(i),\nu(i)}\}_{i \in \mathbb{N}}$  such that  $\alpha(k_{n(i),\nu(i)}) \rightarrow 0$  for  $i \rightarrow \infty$ . None of the classes  $k_{n(i),\nu(i)}$  can be the zero class, since  $(\overline{M}, \overline{g})$  is causal. Therefore  $\alpha^{-1}(0) \cap \mathfrak{T}$  contains an integer class which is impossible by the assumptions.  $\square$

The initial idea of Mather theory is to shift the focus from geodesics which lift to minimal geodesics in the Abelian cover (minimizers) over to measures on the tangent bundle, invariant under the geodesic flow, which minimize an energy-functional among all invariant Borel measures. Fundamental to this point of view is the completeness of the geodesic flow. In most cases however, even if  $(M, g)$  is compact or class A, the geodesic flow of  $(M, g)$  will not be causally complete (complete Lorentzian manifolds are rare). Therefore an attempt to describe the relationships between the qualitative behavior of maximal causal geodesics and the convexity properties of the stable time separation  $\mathfrak{l}$  using the geodesic flow of  $(M, g)$  is not possible. One could argue to continue to use the one point compactification  $P := TM \cup \{\infty\}$  of  $TM$ , as described in [11], and extend the geodesic flow to  $P$  by setting  $\Phi(\infty, t) \equiv \infty$ . This encounters the following problem. In the presence of incomplete geodesics, some invariant measures will concentrate at  $\infty$ , even though they arise as limit measures of geodesics. Then it is not clear how to define the action of these measures. We circumvent this problem by reparameterizing the geodesic flow of

$(M, g)$  to a flow  $\Phi$  in a way that every flowline remains in a compact part of  $TM$ . Additionally  $\Phi$  satisfy other necessary properties, such as conservatism.

For  $v \in TM$  denote with  $\gamma_v: (\alpha_v, \omega_v) \rightarrow M$  the unique inextendible geodesic of  $(M, g)$  with  $\dot{\gamma}_v(0) = v$ .

**Proposition 4.12.** *Let  $(M, g)$  be a pseudo-Riemannian manifold,  $\Phi^g$  its geodesic flow and  $g_R$  a complete Riemannian metric on  $M$ . Define*

$$\Phi: TM \times \mathbb{R} \rightarrow TM, (v, t) \mapsto \hat{\gamma}'_v(t),$$

where  $\hat{\gamma}_v$  is the tangent field to the constant  $g_R$ -arclength parameterization of  $\gamma_v$  with  $|\hat{\gamma}'_v| = |v|$ . Then  $\Phi$  is a smooth flow, called the pregeodesic flow of  $(M, g)$  relative to  $g_R$ .

*Proof.* Denote with  $\nabla^g$  the Levi-Civita connection of  $(M, g)$  and abbreviate with  $\nabla^R := \nabla^{g_R}$  the Levi-Civita connection of  $(M, g_R)$ . Define the tensor  $T^{g, g_R} := \nabla^g - \nabla^R$ . Let  $0 \neq v \in TM$  and consider the unique  $g$ -geodesic  $\gamma_v: (\alpha_v, \omega_v) \rightarrow M$  with  $\dot{\gamma}_v(0) = v$ . Denote with  $\hat{\gamma}_v: \mathbb{R} \rightarrow M$  the constant  $g_R$ -arclength parameterization of  $\gamma_v$  with  $|\hat{\gamma}'_v| \equiv |v|$  ( $\hat{\gamma}'_v := \frac{d}{dt}\hat{\gamma}_v$ ). Then we have ( $\dot{\gamma}_v = \frac{|\dot{\gamma}_v|}{|v|}\hat{\gamma}'_v$ )

$$\begin{aligned} 0 &= \nabla_{\dot{\gamma}_v}^g \dot{\gamma}_v = \nabla_{\hat{\gamma}'_v}^R \dot{\gamma}_v + T^{g, g_R}(\hat{\gamma}'_v, \dot{\gamma}_v) \\ &= \frac{|\dot{\gamma}_v|^2}{|v|^2} \left( \nabla_{\hat{\gamma}'_v}^R \hat{\gamma}'_v + T^{g, g_R}(\hat{\gamma}'_v, \hat{\gamma}'_v) - \frac{1}{|v|^2} g_R(T^{g, g_R}(\hat{\gamma}'_v, \hat{\gamma}'_v), \hat{\gamma}'_v) \hat{\gamma}'_v \right). \end{aligned}$$

Consequently  $\hat{\gamma}$  satisfies the following ODE of second order:

$$(1) \quad \nabla_{\hat{\gamma}'_v}^R \hat{\gamma}'_v = \frac{1}{|v|^2} g_R(T^{g, g_R}(\hat{\gamma}'_v, \hat{\gamma}'_v), \hat{\gamma}'_v) \hat{\gamma}'_v - T^{g, g_R}(\hat{\gamma}'_v, \hat{\gamma}'_v)$$

It is easy to see that  $g_R(\hat{\gamma}'_v, \hat{\gamma}'_v)$  is preserved along  $\hat{\gamma}_v$ . Equation (1) extends smoothly to  $TM$  and therefore defines a smooth complete flow  $\Phi: TM \times \mathbb{R} \rightarrow TM$ .  $\square$

Note that it is not clear whether for a general spacetime  $(M, g)$  the pregeodesic flow  $\Phi: TM \times \mathbb{R} \rightarrow TM$  is induced by a variational principle. In special cases though this can be the case, for example if  $g_R$  is a first integral of  $\Phi^g$ . The assumption of a variational principle leading to  $\Phi$  is similar to the problem of geodesically equivalent manifolds (see for example [12]).

As we have seen in the proof above, the pregeodesic flow is conservative. A flow  $\Phi: U \subseteq TM \times \mathbb{R} \rightarrow TM$  is called conservative if

$$\frac{d}{dt}(\pi \circ \Phi(v, t)) = \Phi(v, t)$$

for all  $(v, t) \in U$ . This property is of course equivalent to  $\Phi$  being defined by a second order ODE.

From this point on we will not consider  $\Phi$  itself, but the restriction of  $\Phi$  to the unit tangent bundle  $T^{1, R}M$  of  $(M, g_R)$ . We omit the indication of the restriction and denote  $\Phi|_{T^{1, R}M \times \mathbb{R}}$  with  $\Phi$  as well.

Let  $f: M \rightarrow \mathbb{R}$  be a Lipschitz continuous function. For a  $C^1$ -curve  $\gamma: I \rightarrow M$  the composition  $f \circ \gamma: I \rightarrow \mathbb{R}$  is differentiable almost everywhere. Let  $v \in T^{1, R}M$  and  $\gamma: I \rightarrow M$  be a curve tangential to  $v$  in  $s \in I$ . Then the existence and the value of  $\frac{d}{dt}|_{t=s}(f \circ \gamma)$  doesn't depend on  $\gamma$ . Therefore we can define

$$\text{Def}(\partial f) := \{v \in T^{1, R}M \mid \exists \gamma \text{ a curve with } \dot{\gamma}(0) = v \text{ s.t.}$$

$$\lim_{t \rightarrow 0} \frac{f \circ \gamma(t) - f \circ \gamma(0)}{t} =: \partial_v f(v) \text{ exists} \}.$$

By Rademacher's theorem every Lipschitz function is differentiable almost everywhere. Denote the set of points where  $f$  is differentiable with  $\text{Def}(df)$ . Since we have  $TM_p \subseteq \pi_{TM}^{-1}(\text{Def}(df))$  for all  $p \in \text{Def}(df)$  we know that  $\pi_{TM}^{-1}(\text{Def}(df))$  is a

Borel set of full Lebesgue measure. Further, since we have  $\pi_{TM}^{-1}(\text{Def}(df)) \subseteq \text{Def}(\partial f)$  and the Lebesgue measure is complete, we obtain that  $\text{Def}(\partial f)$  is a Borel set of full Lebesgue measure. Define the *partial differential*  $\partial f$  of  $f$  as

$$\partial_v f := \begin{cases} \partial_v f, & \text{for } v \in \text{Def}(\partial f), \\ 0, & \text{else.} \end{cases}$$

$\partial f$  is a bounded measurable function on  $T^{1,R}M$ .

**Lemma 4.13.** *Let  $f: M \rightarrow \mathbb{R}$  be a Lipschitz continuous function and  $\mu$  a finite  $\Phi$ -invariant Borel measure on  $T^{1,R}M$ . Then we have*

$$\int \partial f d\mu = 0.$$

*Proof.* The proof is an application of Fubini's theorem and the conservative property of the pregeodesic flow.  $\square$

Lemma 4.13 permits us to associate a unique homology class to every finite  $\Phi$ -invariant Borel measure  $\mu$  on  $T^{1,R}M$ .

**Definition 4.14.** *For a finite,  $\Phi$ -invariant Borel measure  $\mu$ , define the unique homology class  $\rho(\mu) \in H_1(M, \mathbb{R})$ , satisfying*

$$\langle [\omega], \rho(\mu) \rangle := \int_{T^{1,R}M} \omega d\mu,$$

for every closed 1-form  $\omega$  on  $M$ .

The goal is now to maximize a functional over the set of finite invariant measures with fixed homology class. Like in the case of curves this is sensible only in the class of finite invariant measures with support entirely in the set of future pointing causal vectors. Consequently we consider finite  $\Phi$ -invariant (or for short invariant) Borel measures with support in the set of future pointing vectors of  $T^{1,R}M$ . Denote by  $\mathfrak{M}_g$  the set of such measures.  $\mathfrak{M}_g$  is a cone over  $\mathfrak{M}_g^1$ , the set of invariant probability measures with support in the future pointing  $g_R$ -unit vectors.

**Lemma 4.15.** *For  $(M, g)$  of class A we have  $\mathfrak{T} = \rho(\mathfrak{M}_g)$ .*

*Proof.*  $\rho(\mathfrak{M}_g) \subseteq \mathfrak{T}$ : Let  $\mu \in \mathfrak{M}_g$ . There exists a sequence of positive, finite combinations  $\sum_i \lambda_{i,n} \mu_{i,n}$  of  $\Phi$ -ergodic probability measures  $\mu_{i,n}$  approximating  $\mu$  in the weak-\* topology. Since these combinations are positive, the  $\mu_{i,n}$  are supported in the future pointing vectors as well. Choose  $\mu_{i,n}$ -generic pregeodesics  $\gamma_{i,n}$ . By the Birkhoff ergodic theorem we have

$$\frac{1}{2T}(\gamma_{i,n})_{\#}(\mathcal{L}^1|_{[-T,T]}) \xrightarrow{*} \mu_{i,n}$$

for  $T \rightarrow \infty$ . Consequently  $\mu$  is approximated by

$$\sum_i \frac{\lambda_{i,n}}{2T}(\gamma_{i,n})_{\#}(\mathcal{L}^1|_{[-\frac{T}{\lambda_{i,n}}, \frac{T}{\lambda_{i,n}}]})$$

in the weak-\* topology for  $n, T \rightarrow \infty$ . Choose future pointing curves of length less than  $\text{fill}(g, g_R)$  connecting  $\gamma_{i,n}(\frac{T}{\lambda_{i,n}})$  with  $\gamma_{i+1,n}(-\frac{T}{\lambda_{i+1,n}})$ . Joining these curves in the obvious manner defines a sequence of future pointing curves  $\zeta_{n,T}: [-\bar{T}, \bar{T}] \rightarrow M$  such that  $\frac{1}{2\bar{T}}(\zeta_{n,T})_{\#}(\mathcal{L}^1|_{[-\bar{T}, \bar{T}]})$  approximates  $\sum_i \lambda_{i,n} \mu_{i,n}$  in the weak-\* topology ( $\bar{T} := \sum_i \frac{T}{\lambda_{i,n}}$ ). Since  $\rho(\zeta_{n,T}) \rightarrow \rho(\mu)$  for  $n \rightarrow \infty$  and for an appropriate choice of  $T_n \rightarrow \infty$ , and since  $\mathfrak{T}$  is closed, the rotation vector of  $\mu$  will be contained in the stable time cone.

$\mathfrak{T} \subseteq \rho(\mathfrak{M}_g)$ : Let  $\gamma_n: [-T_n, T_n] \rightarrow M$  be a sequence of future pointing curves and  $C \in [0, \infty)$  with  $C\rho(\gamma_n) \rightarrow h \in \mathfrak{T}$ . Choose a future pointing pregeodesic

$\zeta_n: [-\overline{T}_n, \overline{T}_n] \rightarrow M$  homotopic with fixed endpoints to  $\gamma_n$ . Further choose  $\overline{C}_n \in [0, \infty)$  such that  $\overline{C}_n \rho(\zeta_n) = C \rho(\gamma_n)$ . The sequence  $\{\overline{C}_n\}_{n \in \mathbb{N}}$  is bounded by corollary 2.12. Set  $\mu_n := \frac{\overline{C}_n}{2\overline{T}_n}(\zeta_n)_\#(\mathcal{L}^1|_{[-\overline{T}_n, \overline{T}_n]})$ . Then a subsequence of  $\{\mu_n\}$  converges in the weak-\* topology to a finite invariant Borel measure  $\mu$  with  $\rho(\mu) = h$ . By construction the support of  $\mu$  is a subset of the future pointing  $g_R$ -unit vectors.  $\square$

For  $\mu \in \mathfrak{M}_g$  define  $\mathfrak{L}(\mu) := \int_{T^{1,R}M} \sqrt{-g(v,v)} d\mu(v)$  the *average length* of  $\mu$ . Note that  $\mathfrak{L}$  and  $\omega \mapsto \int \omega d\mu$  for  $\omega \in \Lambda^1(T^*M)$  are continuous functionals on  $\mathfrak{M}_g$  provided with the weak-\* topology.

**Proposition 4.16.** *For  $(M, g)$  class A we have*

$$\mathfrak{l}(h) = \sup\{\mathfrak{L}(\mu) \mid \mu \in \mathfrak{M}_g \text{ with } \rho(\mu) = h \in \mathfrak{T}\}.$$

*Proof.* Clear from above.  $\square$

**Lemma 4.17.** *Let  $(M, g)$  be of class A. Then the set*

$$\rho^{-1}(h) \subseteq \mathfrak{M}_g \subseteq (C^0(T^{1,R}M), \|\cdot\|_\infty)'$$

*is bounded for every  $h \in \mathfrak{T}$ .*

*Proof.* Assume that  $\{\mu \in \mathfrak{M}_g \mid \rho(\mu) = h\}$  is unbounded. Then there exists a sequence of probability measures  $\mu_n \in \mathfrak{M}_g$  with  $\rho(\mu_n) \rightarrow 0$  for  $n \rightarrow \infty$ . Like in the proof to lemma 4.15 we can choose a convex combination  $\sum \lambda_{i,n} \mu_{i,n}$  of ergodic probability measures  $\mu_{i,n}$  approximating  $\mu_n$  in the weak-\* topology. Since  $\mathfrak{T}$  contains no nontrivial linear subspaces (theorem 2.11 (ii)), there exists a sequence of ergodic probability measures  $\mu_{i_n,n}$  with  $\rho(\mu_{i_n,n}) \rightarrow 0$  for  $n \rightarrow \infty$ . Choose for every  $n \in \mathbb{N}$  a  $\mu_{i_n,n}$ -generic pregeodesic  $\gamma_n: \mathbb{R} \rightarrow M$  and  $T_n > 0$  such that

$$\|\rho(\gamma_n|_{[-T_n, T_n]}) - \rho(\mu_{i_n,n})\| \leq \frac{1}{n}.$$

Therefore we have constructed an admissible sequence of future pointing curves whose rotation vectors converge to 0. This contradicts theorem 2.11 (ii), since in this case  $\mathfrak{T}^1$  is not disjoint from  $0 \in H_1(M, \mathbb{R})$ .  $\square$

**Corollary 4.18.** *For every  $h \in \mathfrak{T}$  there exists a maximal measure  $\mu \in \mathfrak{M}_g$ , i.e.  $\mathfrak{L}(\mu) = \mathfrak{l}(\rho(\mu))$ .*

*Proof.* Use lemma 4.17 and the fact that  $\mathfrak{L}$  as well as  $\rho$  are continuous with respect to the weak-\* topology.  $\square$

After we established the existence of maximal invariant measures of  $\Phi$  we can ask about the multiplicity of maximal ergodic measures. Recall  $b := \dim H_1(M, \mathbb{R})$ .

**Proposition 4.19.** *Let  $(M, g)$  be of class A. Then the pregeodesic flow admits at least  $b$ -many maximal ergodic measures.*

*Proof.* Let  $\alpha \in (\mathfrak{T}^*)^\circ$  and consider the subgraph  $\Gamma$  of the restriction  $\mathfrak{l}|_{\alpha^{-1}(1) \cap \mathfrak{T}}$ . Choose an extremal point  $(h, \mathfrak{l}(h))$  of  $\Gamma$  and consider  $\lambda_0 > 0$  maximal among all  $\lambda > 0$  with  $(\rho(\mu), \mathfrak{L}(\mu)) = \lambda(h, \mathfrak{l}(h))$  for some  $\mu \in \mathfrak{M}_g^1$ . The preimage of  $\lambda_0(h, \mathfrak{l}(h))$  under the map  $\mu \in \mathfrak{M}_g^1 \mapsto (\rho(\mu), \mathfrak{L}(\mu))$  is a compact and convex subset of  $\mathfrak{M}_g^1$ . Therefore it contains extremal points by the theorem of Krein-Milman. We want to show that these extremal points are extremal points of  $\mathfrak{M}_g^1$  as well. Assume that there exists an extremal point  $\mu$  of  $\{\nu \in \mathfrak{M}_g^1 \mid (\rho(\nu), \mathfrak{L}(\nu)) = \lambda_0(h, \mathfrak{l}(h))\}$  that is not an extremal point of  $\mathfrak{M}_g^1$ . Then there exist  $\nu_0, \nu_1 \in \mathfrak{M}_g^1$  and  $\eta \in (0, 1)$  with  $\mu = (1 - \eta)\nu_0 + \eta\nu_1$ . In this case both  $\nu_0$  and  $\nu_1$  are maximal since  $\mu$  is maximal. We have  $\rho(\nu_{0,1}) \notin \text{pos}\{\rho(\mu)\}$  since else  $\mathfrak{L}(\mu)$  or  $\lambda_0$  would not be maximal. More precisely we know that either both  $\rho(\nu_0)$  and  $\rho(\nu_1) \in \text{pos}\{\rho(\mu)\}$  or  $\rho(\nu_0)$  and  $\rho(\nu_1) \notin$

$\text{pos}\{\rho(\mu)\}$ . If  $\rho(\nu_0), \rho(\nu_1) \in \text{pos}\{\rho(\mu)\}$  we can choose  $\eta_0, \eta_1 \leq 1$  with  $\rho(\nu_i) = \eta_i \rho(\mu)$  since  $\lambda_0$  was chosen maximal. But then we'd have  $\eta_0 = \eta_1 = 1$  and  $\nu_0, \nu_1 \in \{\nu \in \mathfrak{M}_g^1 \mid \rho(\nu) = \rho(\mu)\}$ . Therefore we have  $\nu_0, \nu_1 \in \{\nu \in \mathfrak{M}_g^1 \mid (\rho(\nu), \mathfrak{L}(\nu)) = \lambda_0(h, \mathfrak{l}(h))\}$  and a contradiction to the assumption follows that  $\mu$  is an extremal point of that set.

In the other case  $\rho(\nu_0), \rho(\nu_1) \notin \text{pos}\{\rho(\mu)\}$  we have

$$\text{pos}\{\text{conv}\{(\rho(\nu_0), \mathfrak{L}(\nu_0)), (\rho(\nu_1), \mathfrak{L}(\nu_1))\}\} \subseteq \text{graph}(\mathfrak{l}).$$

This contradicts our assumption that  $(h, \mathfrak{l}(h))$  is an extremal point of the subgraph of  $\mathfrak{l}|_{\alpha^{-1}(1)}$ . Thus any extremal point of  $\{\nu \in \mathfrak{M}_g^1 \mid (\rho(\nu), \mathfrak{L}(\nu)) = \lambda_0(h, \mathfrak{l}(h))\}$  is an extremal point of  $\mathfrak{M}_g^1$ .

It is well known that the extremal points of  $\mathfrak{M}_g^1$  are ergodic measures. In this case they are maximal ergodic measures. Choose one maximal ergodic measure for every extremal point of the subgraph of  $\mathfrak{l}|_{\alpha^{-1}(1)}$ . The only point left to note is that  $\Gamma$  contains at least  $b$ -many extremal points. This shows our claim.  $\square$

## 5. CALIBRATIONS

Calibrations are a common notion in differential geometry and variational analysis (see [9]). Especially in the calculus of variations they provide a powerful tool to study minimizers of convex variational problems. Since we are solely interested in the case of curves, the general definition of a calibration (in terms of geometric measure theory) is not needed. References for calibrations in the case of curves are [6] and [3]. In [6] calibrations appear as “generalized coordinates”.

To our knowledge the first appearance of calibrations in pseudo-Riemannian geometry is [13]. Therein a calibration is defined as follows. Let  $(M, g)$  be a pseudo-Riemannian manifold and  $A$  a subset of the Grassmann bundle of oriented  $k$ -tangent planes to  $M$ . Then a calibration on  $M$  with respect to  $A$  is defined as a closed differential  $k$ -form  $\varphi$  such that  $\varphi(\xi) \geq \text{vol}(\xi)$  for all  $\xi \in A$  where  $\text{vol}$  denotes the  $k$ -volume relative to  $g$ . This definition is inspired by the definition of a calibration in [9] for Riemannian manifolds.

To obtain the existence of calibrations in the pseudo-Riemannian category, even in the most simple cases, it is necessary to restrict the condition to a subset of all tangent planes. More precisely consider  $\mathbb{R}_n^m := (\mathbb{R}^m, \langle \cdot, \cdot \rangle_n)$ , where  $\langle \cdot, \cdot \rangle_n$  is a symmetric inner product of signature  $(n, m-n)$ . Then define  $G^r(p, \mathbb{R}_n^m)$  to be the set of all  $p$ -dimensional linear subspaces  $\xi$  of  $\mathbb{R}^m$  such that  $\langle \cdot, \cdot \rangle_n|_{\xi \times \xi}$  is nondegenerate, and

$$G(k, l, \mathbb{R}_n^m) := \{\xi \in G^r(k+l, \mathbb{R}^m) \mid \text{ind}(\langle \cdot, \cdot \rangle_n|_{\xi \times \xi}) = k\}.$$

Then Mealy made the following observations:

**Observation 1** ([13]).  $G(k, l, \mathbb{R}_n^m)$ , with  $n, m-n > 0$  and  $(k, l) \notin \{(n, 0), (0, m-n), (n, m-n)\}$  can not support an inequality  $\varphi(\xi) \geq 1$  for all  $\xi$  in any connected component of  $G(k, l, \mathbb{R}_n^m)$ .

**Observation 2** ([13]).  $G(k, l, \mathbb{R}_n^m)$ , with  $n, m-n > 0$  and  $(k, l) \notin \{(n, 0), (0, m-n), (n, m-n)\}$  can not support an inequality of the following form:  $\varphi(\xi) \leq 1$  for all  $\xi$  in any connected component of  $G(k, l, \mathbb{R}_n^m)$ , such that there exists  $\xi$  in this component with  $\varphi(\xi) = 1$ .

The connected components of  $G(k, l, \mathbb{R}_n^m)$  are a natural choice for  $A$  since they are the natural constraint for the tangents spaces of submanifolds one would call causally constant. A smooth connected submanifold  $N$  of  $M$  is causally constant if  $g|_{TN \times TN}$  is either positive or negative semidefinite on all of  $N$ . Note that for curves in a spacetime this is equivalent to being future or past pointing. Both



observations together show that the only dimensions, where one could expect calibrated submanifolds to exist, are  $n$  and  $m - n$ , i.e. causally constant submanifolds of maximal dimension.

To obtain the full analog of the definition of a calibration one has to impose the condition that the infimum of 1 is actually attained, i.e.  $\inf_{\xi \in A} \varphi(\xi) = 1$ . This notion of calibration is closer to the one in [9]. Now a calibration in the case  $k = 1$  would be a closed 1-form  $\varphi$  with  $\inf\{\varphi(v) \mid v \in \text{Time}(M, [g]), g(v, v) = -1\} = 1$ . Note that the existence problem for this definition has i.g. no solution in the smooth category. We will not follow this strategy, but rather proceed as in [6] and [3]. It will be an easy consequence of the results below that the calibrations induce (bounded, measurable) calibrations in a weak version of the above sense.

Consider a compact spacetime  $(M, g)$  with Lorentzian cover  $(M', g')$ . Let  $l \in (0, \infty)$ . We call a function  $\tau: M' \rightarrow \mathbb{R}$  an  $l$ -pseudo-time function if for every  $p' \in M'$  there exists a convex normal neighborhood  $U$  of  $p'$  such that

$$\tau(q') - \tau(p') \geq l d(p', q')$$

for all  $q' \in J_U^+(p')$ . Note that if  $\tau$  is Lipschitz, the inequality  $\tau(q') - \tau(p') \geq l d(p', q')$  already implies that  $\tau$  is a time function. This is due to the non-Lipschitz continuity of the time separation on the boundary  $\partial(J_U^+(p'))$  for any  $p' \in M'$ .

**Lemma 5.1.** *Let  $(M, g)$  be a compact spacetime and  $(M', g')$  a Lorentzian cover. Further let  $l, L \in (0, \infty)$  and  $\tau: M' \rightarrow \mathbb{R}$  be a  $L$ -Lipschitz  $l$ -pseudo-time function of  $(M', g')$ . Then there exists  $\varepsilon = \varepsilon(l, L) > 0$  such that*

$$\tau(q') - \tau(p') \geq \varepsilon \text{dist}(p', q')$$

for all  $p', q' \in M'$  with  $q' \in J^+(p')$ .

Recall the definition of  $\text{Def}(\partial f)$  for a Lipschitz function  $f$ . Then lemma 5.1 implies  $\partial_v \tau \geq \varepsilon |v|$  for all future pointing  $v \in \text{Def}(\partial \tau)$ . We obtain the following corollary for the almost everywhere defined total differential of  $\tau$ .

**Corollary 5.2.** *Under the assumptions of lemma 5.1 we have*

$$-d\tau_{p'}^\# \in \text{Time}(M', [g'])^{\varepsilon'}$$

for some  $\varepsilon' > 0$ , whenever  $d\tau_{p'}$  exists.

*Proof of lemma 5.1.* Lift  $g_R$  to a Riemannian metric  $g'_R$  on  $M'$ . Let  $p', q' \in M'$ . We can assume that  $\text{dist}(p', q')$  to be as small as we wish. Just observe that for  $q' \in J^+(r')$  and  $r' \in J^+(p')$  with  $\tau(q') - \tau(r') \geq \varepsilon \text{dist}(r', q')$  and  $\tau(r') - \tau(p') \geq \varepsilon \text{dist}(p', r')$ , we have

$$\begin{aligned} \tau(q') - \tau(p') &= \tau(q') - \tau(r') + \tau(r') - \tau(p') \geq \varepsilon \text{dist}(r', q') + \varepsilon \text{dist}(p', r') \\ &\geq \varepsilon \text{dist}(p', q'). \end{aligned}$$

Consequently we can assume that  $p'$  and  $q'$  are contained in a convex normal neighborhood  $U$  such that  $\partial(J_U^+(p')) \cap \partial(J_U^-(q')) \neq \emptyset$ , i.e.  $q' \in J_U^+(p')$ . Under this assumption it is sufficient to prove the claim for  $q' \in \partial(J_U^+(p'))$ . Note that for  $q'' \in \partial(J_U^+(p')) \cap \partial(J_U^-(q'))$  we have  $\text{dist}(p', q'') \geq \frac{1}{2} \text{dist}(p', q')$  or  $\text{dist}(q'', q') \geq \frac{1}{2} \text{dist}(p', q')$ . Then we get

$$\tau(q') - \tau(p') \geq \tau(q'') - \tau(p') \geq \varepsilon \text{dist}(p', q'') \geq \frac{\varepsilon}{2} \text{dist}(p', q')$$

if  $\text{dist}(p', q'') \geq \frac{1}{2} \text{dist}(p', q')$ . In the other case we get

$$\tau(q') - \tau(p') \geq \tau(q') - \tau(q'') \geq \varepsilon \text{dist}(q'', q') \geq \frac{\varepsilon}{2} \text{dist}(p', q').$$

Further it suffices to consider the case  $\text{dist}(p', q'') \geq \frac{1}{2} \text{dist}(p', q')$ , since the other case follows from this one by reversing the time-orientation and replacing  $\tau$  by  $-\tau$ . Consequently we are done if we prove the claim for  $p', q' \in M'$  such that there exists a convex normal neighborhood  $U$  of  $p', q'$  and  $q' \in \partial(J_U^+(p'))$ .

With the local equivalence of Riemannian metrics, this reduces the problem to the vector space  $TM'_{p'}$  together with the Lorentzian metric  $g'_{p'}$  and Riemannian metric  $(g'_R)_{p'}$ . Since any two scalar products on  $TM'_{p'}$  are equivalent, we can assume that  $(TM'_{p'}, g'_{p'}, (g'_R)_{p'})$  is isometric to  $(\mathbb{R}^m, \langle \cdot, \cdot \rangle_1, \langle \cdot, \cdot \rangle_0)$ , where  $\langle \cdot, \cdot \rangle_1 := -(e_0^*)^2 + \sum_{i=1}^{m-1} (e_i^*)^2$  and  $\langle \cdot, \cdot \rangle_0 := \sum_{i=0}^{m-1} (e_i^*)^2$  for the dual basis  $\{e_0^*, \dots, e_{m-1}^*\}$  of the standard basis  $\{e_0, \dots, e_{m-1}\}$  of  $\mathbb{R}^m$ . We can further assume that  $e_0$  is future pointing by applying the isometry  $(\lambda^0, \dots, \lambda^{m-1}) \mapsto (-\lambda^0, \lambda^1, \dots, \lambda^{m-1})$  of  $(\mathbb{R}^m, \langle \cdot, \cdot \rangle_1, \langle \cdot, \cdot \rangle_0)$  if necessary. Denote the set of lightlike future pointing vectors in  $(\mathbb{R}^m, \langle \cdot, \cdot \rangle_1)$  with  $\text{Light}_m$ . Set  $|v|_i := \sqrt{|\langle v, v \rangle_i|}$ , for  $i = 0, 1$ . Now the claim is equivalent to the following problem. Given  $l', L' \in (0, \infty)$ , an open star-shaped neighborhood  $U$  of  $0 \in \mathbb{R}^m$  and a  $L'$ -Lipschitz function  $\tau': U \rightarrow \mathbb{R}$  with  $\tau'(w) - \tau'(0) \geq l'|w|_1$  for all future pointing vectors  $w \in U$ . Then there exists  $\varepsilon' = \varepsilon'(l', L') > 0$  such that  $\tau'(v) - \tau'(0) \geq \varepsilon'|v|_0$  for all  $v \in \text{Light}_m \cap U$ .

Let  $v \in \text{Light}_m$  be given. Define  $N: \text{Light}_m \rightarrow \mathbb{R}^m$  to be the Euclidian unit normal to the light cone with  $e_0^* \circ N(\cdot) > 0$ . Note that  $N(v) \in \text{Light}_m$  and  $\langle v, N(v) \rangle_1 = -|v|_0$  for all  $v \in \text{Light}_m$  for our choice of  $\langle \cdot, \cdot \rangle$  and  $\langle \cdot, \cdot \rangle_1$ . Then for  $\beta_1, \beta_2 \geq 0$  we have  $|\beta_1 v + \beta_2 N(v)|_1 = \sqrt{\beta_1 \beta_2 |v|_0}$  and

$$\text{dist}_0(\beta_1 v + \beta_2 N(v), \text{Light}_m) = \min\{\beta_1 |v|_0, \beta_2\},$$

where  $\text{dist}_0$  denotes the distance relative to  $|\cdot|_0$ . For  $\beta_2(L', v) := \left(\frac{l'}{2L'}\right)^2 |v|_0$  we have

$$\begin{aligned} |v + \beta_2(L', v)N(v)|_1 &= \sqrt{|v|_0 \left(\frac{l'}{2L'}\right)^2 |v|_0} \geq \frac{2L'}{l'} \min\left\{|v|_0, \left(\frac{l'}{2L'}\right)^2 |v|_0\right\} \\ &= \frac{2L'}{l'} \text{dist}_0(v + \beta_2(L', v)N(v), \text{Light}_m). \end{aligned}$$

Then we have

$$\begin{aligned} \tau'(v) &\geq \tau'(v + \beta_2(L', v)N(v)) - L' \text{dist}_0(v + \beta_2(L', v)N(v), \text{Light}_m) \\ &\geq \tau'(0) + l'|v + \beta_2(L', v)N(v)|_1 - L' \text{dist}_0(v + \beta_2(L', v)N(v), \text{Light}_m) \\ &\geq \tau'(0) + L' \text{dist}_0(v + \beta_2(L', v)N(v), \text{Light}_m) \\ &= \tau'(0) + L' \min\left\{1, \left(\frac{l'}{2L'}\right)^2\right\} |v|_0 =: \tau'(0) + \varepsilon'|v|_0. \end{aligned}$$

Let  $p' \in M'$  be given. Choose a convex normal neighborhood  $U$  of  $p'$  and  $V \subseteq TM'_{p'}$  such that  $\exp_{p'}^{g'}|_V: V \rightarrow U$  is a diffeomorphism. Set  $\tau' := \tau \circ \exp_{p'}^{g'}|_V$ . Since  $M'$  is the cover of the compact manifold  $M$ , there exists a constant  $L' = L'(L) < \infty$ , independent of  $p'$ , such that  $\tau'$  is  $L'$ -Lipschitz. Note that  $\tau'$  is a  $l$ -pseudo time function. This finishes the proof.  $\square$

Recall that a function  $\tau: \overline{M} \rightarrow \mathbb{R}$  is said to be  $\alpha$ -equivariant if  $\tau$  is equivariant under the action of  $H_1(M, \mathbb{Z})$  on  $\overline{M}$  and of  $\alpha(H_1(M, \mathbb{Z}))$  on  $\mathbb{R}$ , i.e.  $\tau(x + k) = \tau(x) + \alpha(k)$  for all  $k \in H_1(M, \mathbb{Z})$  and all  $x \in \overline{M}$ .

**Lemma 5.3.** *Let  $(M, g)$  be a compact and vicious spacetime,  $\alpha \in H^1(M, \mathbb{R})$  and  $f: \overline{M} \rightarrow \mathbb{R}$  an  $\alpha$ -equivariant time function of  $(\overline{M}, \overline{g})$ . Then we have  $\alpha \in (\mathfrak{T}^*)^\circ$ .*

*Proof.* It is clear that  $\alpha \in \mathfrak{T}^*$ , since else there would exist a homology class  $h \in H_1(M, \mathbb{Z}) \cap \mathfrak{T}^\circ$  with  $\alpha(h) < 0$ . Then, by proposition 2.13, there exist  $l \in \mathbb{N}$  and a timelike curve  $\gamma: S^1 \rightarrow M$  representing  $l \cdot h$ . Lifting  $\gamma$  to  $\overline{M}$  yields a timelike curve  $\overline{\gamma}: [0, 1] \rightarrow \overline{M}$  with

$$f(\overline{\gamma}(1)) - f(\overline{\gamma}(0)) = l\alpha(h) < 0.$$

This clearly contradicts the property of a time function.

Now assume that  $\alpha \in \partial \mathfrak{T}^*$ . Choose  $h_\alpha \in (\partial \mathfrak{T} \cap \ker \alpha) \setminus \{0\}$  and future pointing curves  $\delta_n: [0, T_n] \rightarrow \overline{M}$  with  $\text{dist}(\delta_n(T_n) - \delta_n(0), \text{span}\{h_\alpha\}) \leq \text{err}(g, g_R)$ . By construction we have  $f(\delta_n(T_n)) - f(\delta_n(0)) \leq K$  for some constant  $K = K(f) < \infty$ . Cut  $\delta_n$  into sub-arcs  $\delta_{n,k}: [0, a_{n,k}] \rightarrow \overline{M}$  with  $L^{g_R}(\delta_{n,k}) \in [\text{inj}(M, g)/2, \text{inj}(M, g)]$ . From this sequence of “short” curves we obtain a subsequence  $\{\delta'_n\}_{n \in \mathbb{N}}$  with

$$f(\delta'_n(a_{n,k})) - f(\delta'_n(0)) \rightarrow 0.$$

Using the compactness of  $M$  and the  $\alpha$ -equivariance of  $f$  we can assume that  $\{\delta'_n(0)\}_{n \in \mathbb{N}}$  is contained in a compact subset of  $\overline{M}$ . Parameterizing  $\delta'_n$  with respect to  $g_R$ -arclength, we deduce that a subsequence converges uniformly to a future pointing curve  $\delta: [0, a] \rightarrow \overline{M}$  with  $f(\delta(a)) - f(\delta(0)) = 0$ . This again contradicts the time function property.  $\square$

Define the *dual stable time separation*

$$\mathfrak{l}^*: \mathfrak{T}^* \rightarrow \mathbb{R}, \alpha \mapsto \inf\{\alpha(h) \mid \mathfrak{l}(h) = 1\}.$$

**Definition 5.4.** Let  $\alpha \in (\mathfrak{T}^*)^\circ$ . A function  $\tau: \overline{M} \rightarrow \mathbb{R}$  is a *calibration representing  $\alpha$*  if  $\tau$  is an  $\alpha$ -equivariant Lipschitz continuous  $\mathfrak{l}^*(\alpha)$ -pseudo time function.

Lemma 5.3 shows that  $\alpha \in (\mathfrak{T}^*)^\circ$  is a necessary condition for the existence of a calibration representing  $\alpha$ . Next we show that it is sufficient as well.

**Proposition 5.5.** Let  $\omega \in \alpha \in (\mathfrak{T}^*)^\circ$  and  $F: \overline{M} \rightarrow \mathbb{R}$  a primitive of  $\overline{\pi}^*(\omega)$ . Then the function

$$\tau_\omega: \overline{M} \rightarrow \mathbb{R}, x \mapsto \liminf_{\substack{y \in J^+(x), \\ \text{dist}(x, y) \rightarrow \infty}} [F(y) - \mathfrak{l}^*(\alpha) d(x, y)]$$

is a calibration representing  $\alpha$ .

*Proof.* By definition we have  $|\alpha(k)| \geq \mathfrak{l}^*(\alpha)\mathfrak{l}(k)$  for any  $k \in H_1(M, \mathbb{Z})_\mathbb{R}$ . Further note that, since  $k$  is an integer class, we get  $\mathfrak{l}(k) \geq d(x, x+k)$  for all  $x \in \overline{M}$ . For  $y \in \overline{M}$  choose  $k \in H_1(M, \mathbb{Z})_\mathbb{R}$  with  $x+k \in J^+(y) \cap B_{\text{fill}(g, g_R)}(y)$ . Then we have  $F(y) \geq F(x+k) - \|\omega\|_\infty \text{fill}(g, g_R)$ . Since by the definition of  $\tau_\omega$  we have  $y \in J^+(x)$ , we conclude  $k \in \mathfrak{T}$ . Consequently we obtain (Note that  $\mathfrak{l}(k) \geq d(x, x+k)$  for all  $x \in \overline{M}$  and  $k \in H_1(M, \mathbb{Z})_\mathbb{R} \cap \mathfrak{T}$ .)

$$\begin{aligned} F(y) - \mathfrak{l}^*(\alpha) d(x, y) &\geq F(x+k) - \|\omega\|_\infty \text{fill}(g, g_R) - \mathfrak{l}^*(\alpha) d(x, x+k) \\ &\geq F(x) + \alpha(k) - \mathfrak{l}^*(\alpha)\mathfrak{l}(k) - \|\omega\|_\infty \text{fill}(g, g_R) \\ &\geq F(x) - \|\omega\|_\infty \text{fill}(g, g_R). \end{aligned}$$

Consequently we have  $\tau_\omega(x) > -\infty$ .

In order to show  $\tau_\omega(x) < \infty$ , consider a homology class  $h \in \{h' \in \mathfrak{l}^{-1}(1) \mid \alpha(h') = \mathfrak{l}^*(\alpha)\}$  and a sequence  $\{\gamma_n: [a_n, b_n] \rightarrow \overline{M}\}_{n \in \mathbb{N}}$  of maximizers with

$$\frac{1}{b_n - a_n} (\gamma_n(b_n) - \gamma_n(a_n), L^g(\gamma_n)) \rightarrow (h, \mathfrak{l}(h)).$$

The existence of  $\gamma_n$  follows from remark 4.6. Choose a sequence  $\varepsilon_n \downarrow 0$  such that

$$\mathfrak{l}^*(\alpha) \left[ \frac{1}{b_n - a_n} L^g(\gamma_n) + \varepsilon_n \right] \geq \alpha(\rho(\gamma_n))$$

for all  $n \in \mathbb{N}$ . Then there exists a sequence of subarcs  $\{\gamma_n|_{[c_n, d_n]}\}_{n \in \mathbb{N}}$  with  $\varepsilon_n(d_n - c_n)\mathfrak{l}^*(\alpha) \leq 1$ ,  $d_n - c_n \rightarrow \infty$  and

$$\mathfrak{l}^*(\alpha) \left[ \frac{1}{d_n - c_n} L^g(\gamma_n|_{[c_n, d_n]}) + \varepsilon_n \right] \geq \alpha(\rho(\gamma_n|_{[c_n, d_n]})).$$

We can assume that  $\gamma_n(c_n) \in J^+(x) \cap B_{\text{fill}(g, g_R)}(x)$ . Choose  $k_n \in H_1(M, \mathbb{Z})_{\mathbb{R}}$  with  $x + k_n \in J^+(\gamma_n(d_n)) \cap B_{\text{fill}(g, g_R)}(\gamma_n(d_n))$ . Then we have  $\|\gamma_n(d_n) - \gamma_n(c_n) - k_n\| \leq 2 \text{fill}(g, g_R) + \text{std}(g_R)$ . Now we can estimate:

$$\begin{aligned} \tau_\omega(x) &\leq \liminf_{n \rightarrow \infty} [F(x + k_n) - \mathfrak{l}^*(\alpha) d(x, x + k_n)] \\ &\leq \liminf_{n \rightarrow \infty} [\alpha(k_n) - \mathfrak{l}^*(\alpha) L^g(\gamma_n|_{[c_n, d_n]})] + F(x) \\ &\leq \liminf_{n \rightarrow \infty} [\alpha(\gamma_n(d_n) - \gamma_n(c_n)) - \mathfrak{l}^*(\alpha) L^g(\gamma_n|_{[c_n, d_n]})] \\ &\quad + F(x) + \|\alpha\| (2 \text{fill}(g, g_R) + \text{std}(g_R)) \\ &\leq F(x) + \|\alpha\| (2 \text{fill}(g, g_R) + \text{std}(g_R)) + 1 < \infty. \end{aligned}$$

Therefore  $\tau_\omega$  is a well defined function on  $\overline{M}$ .

The  $\alpha$ -equivariance of  $\tau_\omega$  follows easily from the  $\alpha$ -equivariance of  $F$ . For  $k \in H_1(M, \mathbb{Z})_{\mathbb{R}}$  we have

$$\begin{aligned} \tau_\omega(x + k) &= \liminf [F(y) - \mathfrak{l}^*(\alpha) d(x + k, y)] \\ &= \liminf [F(y + k) - \mathfrak{l}^*(\alpha) d(x + k, y + k)] \\ &= \liminf [F(y) - \mathfrak{l}^*(\alpha) d(x, y)] + \alpha(k) = \tau_\omega(x) + \alpha(k). \end{aligned}$$

For  $x \in \overline{M}$  consider a sequence of maximizers  $\gamma_n: [a_n, b_n] \rightarrow \overline{M}$  with  $\gamma_n(a_n) = x$  and  $\tau_\omega(x) = \lim_{n \rightarrow \infty} F(\gamma_n(b_n)) - \mathfrak{l}^*(\alpha) L^g(\gamma_n)$ . Note that there exists a constant  $C < \infty$ , independent of  $x$ , such that

$$L^g(\gamma_n|_{[c, d]}) \geq \max\{d(y, z) \mid \alpha(z - y) = \alpha(\gamma_n(d) - \gamma_n(c))\} - C.$$

for any subarc  $\gamma_n|_{[c, d]}$  of  $\gamma_n$ . Since we assumed  $\alpha \in (\mathfrak{T}^*)^\circ$ , the maximum will eventually exceed  $C$ , for  $d - c$  sufficiently large. This immediately shows that every limit pregeodesic of  $\gamma_n$  is timelike. The Lipschitz continuity of  $\tau_\omega$  now follows in the same fashion as in [7].

To see why  $\tau_\omega$  is a  $\mathfrak{l}^*(\alpha)$ -pseudo time function let  $x \in \overline{M}$  and  $x' \in J^+(x)$  be given. Then the reversed triangle inequality implies

$$\begin{aligned} \tau_\omega(x') &= \liminf_{d(x', y) \rightarrow \infty} F(y) - \mathfrak{l}^*(\alpha) d(x', y) \\ &\geq \liminf_{d(x, y) \rightarrow \infty} [F(y) - \mathfrak{l}^*(\alpha) d(x, y)] + \mathfrak{l}^*(\alpha) d(x, x') \\ &\geq \tau_\omega(x) + \mathfrak{l}^*(\alpha) d(x, x'). \end{aligned}$$

□

It is well known that for a Riemannian manifold  $(M, g_R)$  the co-mass norm  $\|\alpha\|^* := \inf\{\|\omega\|_\infty \mid \omega \in \alpha\}$  on  $H^1(M, \mathbb{R})$  coincides with the dual of the stable norm  $\|\cdot\|$ . Thus a natural question is: Is the analogous result true for the stable time separation and the dual time separation? We want to give a positive answer to this question on  $(\mathfrak{T}^*)^\circ$  and discuss why it is i.g. not possible to extend the result to  $\partial \mathfrak{T}^*$ .

Define

$$|\iota|^g := \begin{cases} \sqrt{|g(\iota^\sharp, \iota^\sharp)|}, & \text{if } -\iota^\sharp \in \overline{\text{Time}(M, [g])}, \\ -\infty, & \text{else} \end{cases}.$$

The Cauchy-Schwarz inequality for Lorentzian inner products reformulates to  $|\iota(v)| \geq |\iota|^g |v|_g$ , whenever  $v$  is future pointing.

**Definition 5.6.** For  $\omega \in \Lambda^1(T^*M)$  define

$$l_\infty(\omega) := \min_{p \in M} \{|\omega_p|^g\} \in \mathbb{R}_{\geq 0} \cup \{-\infty\}.$$

1-forms  $\omega$  with  $l_\infty(\omega) > -\infty$  will be called *future pointing*.

With the Cauchy-Schwarz inequality we have

$$\left| \int_a^b \omega_{\gamma(t)}(\dot{\gamma}(t)) dt \right| \geq l_\infty(\omega) L^g(\gamma)$$

for any future pointing curve  $\gamma: [a, b] \rightarrow M$ . This ensures that the function

$$l': H_1(M, \mathbb{R}) \rightarrow \mathbb{R} \cup \{-\infty\}, \quad \alpha \mapsto \sup\{l_\infty(\omega) \mid \omega \in \alpha\}.$$

is well defined. Note that  $|\alpha(k)| \geq l'(\alpha) d(x, x+k)$  for every  $k \in H_1(M, \mathbb{Z})_{\mathbb{R}}$  and any  $x \in \overline{M}$ . It is clear that  $l'(\alpha) > 0$  if and only if  $\alpha$  contains a representative  $\omega \in \Lambda^1(T^*M)$  such that  $-\omega_p^\sharp \in \text{Time}(M, [g])$ . The pullback of  $\omega$  to  $\overline{M}$  is the differential of a  $\alpha$ -equivariant temporal function. The cohomology classes giving rise to a  $\alpha$ -equivariant temporal function were described in theorem 2.11 (iii) by the property  $\alpha^{-1}(0) \cap \mathfrak{T} = \{0\}$ . With theorem 2.11 we obtain the following corollary.

**Corollary 5.7.** Let  $(M, g)$  be of class A and  $\alpha \in H^1(M, \mathbb{R})$ . Then the following statements are equivalent:

- (i)  $l'(\alpha) > 0$
- (ii)  $\alpha \in (\mathfrak{T}^*)^\circ$
- (iii)  $\alpha$  contains a timelike representative, i.e. there exists an  $\omega \in \alpha$  with  $-\omega_p^\sharp \in \text{Time}(M, [g])$  for all  $p \in M$ .
- (iv) There exists  $\omega \in \alpha$  such that  $\pi^*\omega$  is the differential of a smooth temporal function.

**Proposition 5.8.** Let  $(M, g)$  be of class A. Then  $l'$  coincides with the dual function of  $l$  on  $(\mathfrak{T}^*)^\circ$ , i.e.  $l'(\alpha) = l^*(\alpha)$  for all  $\alpha \in (\mathfrak{T}^*)^\circ$ .

*Proof.* (i) Let  $\omega \in \alpha \in \mathfrak{T}^*$  and  $\mu \in \mathfrak{M}^g$ . Then we have

$$\alpha(\rho(\mu)) = \int \omega d\mu \geq l_\infty(\omega) \mathfrak{L}(\mu)$$

and therefore  $\alpha(h) \geq l'(\alpha) l(h)$  for all  $h \in \mathfrak{T}$ . This shows  $l^*(\alpha) \geq l'(\alpha)$  for all  $\alpha \in \mathfrak{T}^*$ .

(ii) Let  $\omega \in \alpha \in (\mathfrak{T}^*)^\circ$ . To show the inequality  $l^*(\alpha) \leq l'(\alpha)$ , we approximate the calibration  $\tau_\omega$ , constructed in proposition 5.5, by primitives of lifts of 1-forms  $o \in \alpha$  to  $\overline{M}$ . Let  $F \in C^\infty(\overline{M})$  be a primitive of  $\pi^*\omega$ . For  $x \in \overline{M}$  choose  $y_n \in \overline{M}$  and maximizers  $\gamma_n$  connecting  $x$  with  $y_n$  such that

$$\tau_\omega(x) = \lim_{n \rightarrow \infty} [F(y_n) - l^*(\alpha) L^g(\gamma_n)].$$

Let  $\gamma$  be any limit curve of  $\{\gamma_n\}_{n \in \mathbb{N}}$ . Then  $\gamma$  maximizes arclength and we have

$$\begin{aligned} \tau_\omega(\gamma(t)) &= \liminf_{d(\gamma(t), y) \rightarrow \infty} [F(y) - l^*(\alpha) d(\gamma(t), y)] \\ &\leq \liminf_{n \rightarrow \infty} [F(y_n) - l^*(\alpha) d(\gamma(t), y_n)] \\ &= \liminf_{n \rightarrow \infty} [F(y_n) - l^*(\alpha) d(\gamma_n(t), y_n)] \\ &= \liminf_{n \rightarrow \infty} [F(y_n) - l^*(\alpha) (d(x, y_n) + L^g(\gamma_n|_{[0,t]}))] \\ &= \tau_\omega(x) + l^*(\alpha) d(x, \gamma(t)) \end{aligned}$$

for all  $t > 0$ . This implies  $\tau_\omega(\gamma(t)) = \tau_\omega(x) + l^*(\alpha) d(x, \gamma(t))$  since  $\tau_\omega$  is a calibration representing  $\alpha$ .

For  $p \in M$  denote with  $\text{inj}(M, g)_p$  the supremum over all  $\eta > 0$  such that  $B_\eta(p)$  is contained in a convex normal neighborhood of  $p$  in  $(M, g)$  with  $g_R$ -diameter at most

1. Define  $\text{inj}(M, g) := \inf\{\text{inj}(M, g)_p \mid p \in M\}$ . Since  $(\overline{M}, \overline{g})$  covers the compact spacetime  $(M, g)$ , we have  $\text{inj}(\overline{M}, \overline{g}) > 0$ .

For a convolution kernel  $\rho$  define

$$\tau_{\omega, \delta}: \overline{M} \rightarrow \mathbb{R}, p \mapsto \delta^{-m} \int_{T\overline{M}_p} \tau_{\omega}(\exp_p^g(v)) \varrho(\delta^{-1}|v|) \text{vol}^g(v)$$

for  $\delta < \text{inj}(\overline{M}, \overline{g})$ . Choose, using corollary 5.2,  $\varepsilon_0 > 0$  such that  $d\tau_{\omega}^{\sharp} \in \text{Time}(\overline{M}, [\overline{g}])^{\varepsilon_0}$ , whenever  $d\tau_{\omega}$  exists. By standard theory we have

$$d\tau_{\omega, \delta}(\cdot) = \delta^{-m} \int d\tau_{\omega} \circ (\exp_p^g)_{*,v}(\cdot) \varrho(\delta^{-1}|v|) \text{vol}^g(v).$$

Since every fibre of  $\text{Time}(\overline{M}, [\overline{g}])^{\varepsilon_0}$  is convex and  $(\exp_p^g)_{*,v} = \text{id}_{T\overline{M}_p}$ , there exist  $\varepsilon_1 > 0$  and  $\delta_1 > 0$  such that

$$-d\tau_{\omega, \delta}^{\sharp} \in \text{Time}(\overline{M}, [\overline{g}])^{\varepsilon_1}$$

for all  $\delta < \delta_1$ . By the calibration property we have  $d\tau_{\omega}(v) \geq \mathbf{l}^*(\alpha)|v|_g$  for all future pointing  $v \in T^{1,R}M$  such that  $d(\tau_{\omega})_{\pi_{TM}(v)}$  exists. Like before we can choose for every  $\varepsilon_2 > 0$  a real number  $\delta_2 = \delta_2(\varepsilon_1, \varepsilon_2) > 0$  such that

$$(2) \quad d(\tau_{\omega, \delta})(v) \geq (1 - \varepsilon_2)\mathbf{l}^*(\alpha)|v|_g$$

for all  $\delta < \delta_2$  and  $v \in \text{Time}(\overline{M}, [\overline{g}])^{\varepsilon_1}$ . The function  $d(\tau_{\omega, \delta})_{\pi(v)}(v)$  attains its minimum exactly at the positive multiples of  $-d(\tau_{\omega, \delta})^{\sharp} \in \text{Time}(\overline{M}, [\overline{g}])^{\varepsilon_1}$ . By the Cauchy-Schwarz inequality for Lorentzian inner products this minimum is a global minimum for all future pointing vectors. Therefore (2) holds for all  $v \in \text{Time}(\overline{M}, [\overline{g}])$  and we have

$$l_{\infty}(d\tau_{\omega, \delta}) \geq (1 - \varepsilon_2)\mathbf{l}^*(\alpha)$$

if  $0 < \delta < \delta_2$ . Recall that  $d\tau_{\omega, \delta}$  is, for  $\delta$  sufficiently small, an  $H_1(M, \mathbb{Z})_{\mathbb{R}}$ -invariant smooth 1-form. It induces a smooth closed 1-form on  $M$  representing  $\alpha$ . Consequently  $\mathbf{l}^*(\alpha)$  is indeed the supremum of the set  $\{l_{\infty}(o)\}_{o \in \alpha}$ .  $\square$

It is easy to construct examples of class A metrics on the 2-torus for which the dual function of  $\mathbf{l}$  does not coincide with  $\mathbf{l}'$  on  $\partial\mathfrak{T}$ . More precisely for these metrics we can show that if  $\alpha \in \partial\mathfrak{T}^* \setminus \{0\}$  we have  $\mathbf{l}'(\alpha) = -\infty$ .

Consider  $\mathbb{R}^2$  together with the standard coordinates  $\{x, y\}$  and standard basis  $\{e_1, e_2\}$ . Choose a  $\mathbb{Z}^2$ -invariant Lorentzian metric  $\overline{g}$  on  $\mathbb{R}^2$  such that the lightlike distributions are generated by  $\overline{X}_1 := -\sin^2(\pi x)\partial_x + \partial_y$  and  $\overline{X}_2 := \partial_x + \sin^2(\pi y)\partial_y$ . Further choose the time-orientation of  $\overline{g}$  such that  $\partial_x$  is future pointing. Finally define the standard scalar product on  $\mathbb{R}^2$  as Riemannian background metric.  $(\mathbb{R}^2, \overline{g})$  induces a class A spacetime structure on  $T^2 := \mathbb{R}^2/\mathbb{Z}^2$ . We have  $\mathfrak{T} = \text{pos}\{e_1, e_2\}$ . Assume that  $\mathbf{l}'(\alpha) \geq 0$  for some  $0 \neq \alpha \in \partial\mathfrak{T} = \text{pos}\{e_1^* \cup \{e_2^*\}$ . Since  $\mathfrak{T}^*$  is a cone, we can assume  $\alpha = e_1^*$ . The other case  $\alpha \sim e_2^*$  follows when exchanging coordinates. Choose  $\omega \in \alpha$  with  $l_{\infty}(\omega) \geq 0$ .

Denote with  $X_1$  the vector field induced by  $\overline{X}_1$  on  $T^2$  and its flow with  $\Phi_1$ . Choose a point  $p \in T^2$  such that  $x(\overline{p}) \notin \mathbb{Z}$  for one (hence every) lift  $\overline{p}$  of  $p$  to  $\mathbb{R}^2$ . Then we have

$$\text{dist}(\Phi_1(p, n), \Phi_1(p, -n)) \rightarrow 0 \text{ for } n \rightarrow \infty$$

and  $\int_{-n}^n \omega(X_1(\Phi_1(p, t)))dt \geq 0$ . Denote with  $\gamma_n$  the shortest Riemannian geodesic connecting  $\Phi_1(p, n)$  with  $\Phi_1(p, -n)$ . The curve  $\zeta_n := \Phi_1(p, \cdot)|_{[-n, n]} * \gamma$  represents the homology class  $2ne_2 - e_1$ . Thus we have  $\int_{\zeta_n} \omega = -1$ . Since  $\int_{\gamma_n} \omega \leq \|\omega\|_{\infty} \text{dist}(\Phi_1(p, n), \Phi(p, -n))$ , we obtain a contradiction for sufficiently large  $n$ .



## 6. CONVEXITY PROPERTIES AND CALIBRATED CURVES

First we introduce *limit measures* of a curve  $\gamma: \mathbb{R} \rightarrow M$ . Consider a  $g_R$ -arclength parameterized  $C^1$ -curve  $\gamma: \mathbb{R} \rightarrow M$ , the continuous tangent curve  $\dot{\gamma}: \mathbb{R} \rightarrow T^{1,R}M$  and a finite Borel measure  $\mu$  on  $T^{1,R}M$ . We call  $\mu$  a *limit measure* of  $\dot{\gamma}$  (or of  $\gamma$ ) if there exist a sequence of closed intervals  $\{[a_i, b_i]\}_{i \in \mathbb{N}}$  with  $b_i - a_i$  diverging to  $\infty$  and a  $C > 0$ , such that  $\frac{C}{b_i - a_i} \dot{\gamma}_\#(\mathcal{L}^1|_{[a_i, b_i]})$  converges to  $\mu$  in the vague topology. Note that the set of limit measures  $\mu$  of a curve  $\gamma$  with  $\mu(T^{1,R}M) \leq C$  is weak-\* compact for all  $C > 0$ .

For  $\alpha \in \mathfrak{T}^*$  we denote with  $\mathfrak{M}_\alpha$  the set of invariant measures which maximize  $\mathfrak{L}_\alpha: \mathfrak{M}_g \rightarrow \mathbb{R}, \mu \mapsto \mathfrak{l}^*(\alpha)\mathfrak{L}(\mu) - \alpha(\rho(\mu))$ . Define  $\text{supp } \mathfrak{M}_\alpha := \cup_{\mu \in \mathfrak{M}_\alpha} \text{supp } \mu$ .

Call a future pointing maximizer  $\gamma: \mathbb{R} \rightarrow M$  a  $\mathfrak{T}^\circ$ -maximizer if there exist  $\lambda_1, \dots, \lambda_{b+1} \geq 0$  and limit measures  $\mu_1, \dots, \mu_{b+1}$  of  $\gamma$  such that  $\rho(\sum \lambda_i \mu_i) \in \mathfrak{T}^\circ$ .

**Proposition 6.1.** *Let  $(M, g)$  be of class A and  $\gamma: \mathbb{R} \rightarrow M$  be a  $\mathfrak{T}^\circ$ -maximizer. Then there exists an  $\alpha \in \mathfrak{T}^*$  such that all limit measures of  $\gamma$  maximize  $\mathfrak{L}_\alpha$ .*

The assumptions of the proposition do not cover all interesting cases. For instance, in the Hedlund examples of section 8, no maximizer  $\gamma: \mathbb{R} \rightarrow \mathbb{R}^3$  satisfies the assumptions. Still every maximizer, asymptotic to the straight lines in  $L$  satisfies the conclusion. In the general case, though, we have to be careful about the maximizing property of the limit measures of a maximizer. Unlike in the positive definite case, limit measures of maximizers need not be maximal. How this fails can be seen again from the Hedlund examples: Every class A spacetime contains lightlike maximizers. Since  $\mathfrak{l} > 0$  on  $\mathfrak{T} \setminus \{0\}$  for the Lorentzian Hedlund examples, no limit measure of a lightlike maximizer maximizes any  $\mathfrak{L}_\alpha$ .

Another complication that appears is that even though all limit measures may maximize  $\mathfrak{L}$ , no  $\alpha \in \mathfrak{T}^*$  exists to satisfy the conclusion of proposition 6.1. For example, consider the flat torus  $(T^n, \langle \cdot, \cdot \rangle_1)$  and a lightlike pregeodesic  $\gamma: \mathbb{R} \rightarrow T^n$  therein.  $\gamma$  is obviously a maximizer. But there exists no  $\alpha \in \mathfrak{T}^*$  such that any limit measure of  $\gamma$  maximizes  $\mathfrak{L}_\alpha$ . Nonetheless, the Hedlund examples show that it is interesting to consider the problem for maximizers whose limit measures have rotation vectors solely contained in the boundary of  $\mathfrak{T}$ . In this case, we have to restrict our considerations to faces of  $\mathfrak{T}$ .

For a maximizer  $\gamma: \mathbb{R} \rightarrow M$  consider the convex hull of all rotation vectors of limit measures of  $\gamma$ . Denote with  $F_\gamma$  the unique face of  $\mathfrak{T}$  of minimal dimension such that the convex hull of all limit measures of  $\gamma$  is contained in  $F_\gamma$ . Then we can use the method of proof for proposition 6.1 and the theorem of Hahn-Banach to obtain the following proposition.

**Proposition 6.2.** *There exists  $\alpha \in \mathfrak{T}^*$  such that all limit measures of  $\gamma$  maximize  $\mathfrak{L}_\alpha|_{\rho^{-1}(F_\gamma)}$  if and only if all convex combinations of limit measures of  $\gamma$  maximize  $\mathfrak{L}$  in their homology class.*

Another notable consequence of proposition 6.1 and the fact that  $\mathfrak{l}$  is positive on  $\mathfrak{T}^\circ$  is the following corollary.

**Corollary 6.3.** *Let  $(M, g)$  be of class A and  $\gamma: \mathbb{R} \rightarrow M$  be a maximizer. Then there exists  $\alpha' \in \mathfrak{T}^*$  such that every limit measure  $\mu$  of  $\gamma$  with vanishing average length is contained in  $\ker(\alpha')$ .*

*Proof of proposition 6.1.* The main idea is taken from the proof of proposition 2 in [11]. Several points need special attention, though. These include the issue of connectivity by future pointing curves. To keep the exposition clear and complete, we present the proof in detail.

Let  $\Sigma_\gamma \subseteq H_1(M, \mathbb{R}) \times \mathbb{R}$  denote the convex hull of the set of pairs  $(\rho(\mu), \mathfrak{L}(\mu))$ , where  $\mu$  is a limit measure of  $\gamma$ . The claim is easily seen to be equivalent to the statement that  $\Sigma_\gamma \subseteq \text{graph } \mathfrak{l}$ .

The idea is to prove  $\Sigma_\gamma \subseteq \text{graph } \mathfrak{l}$  by contradiction. Otherwise, there would exist  $(h, z) \in \Sigma_\gamma$  with  $z < \mathfrak{l}(h)$ . Since  $\gamma$  is a  $\mathfrak{T}^\circ$ -maximizer, we can assume that  $h \in \mathfrak{T}^\circ$ . This can be done by adding a convex combination of limit measures of  $\gamma$  contained in  $\mathfrak{T}^\circ$  to the given convex combination. Since  $\mathfrak{l}$  is concave, this does not alter our assumptions. Consequently, there exist limit measures  $\mu_1, \dots, \mu_l$  of  $\gamma$  and  $\lambda^1, \dots, \lambda^l \geq 0$  with  $\sum \lambda^i = 1$  such that

$$\sum \lambda^i \rho(\mu_i) = h \in \mathfrak{T}^\circ \text{ and } \sum \lambda^i \mathfrak{L}(\mu_i) = z.$$

We can further assume that the limit measures  $\mu_i$  are probability measures. This produces no restriction on the generality of the argument since  $\mathfrak{l}$  is positively homogeneous of degree one.

Choose  $\delta > 0$  with  $h \in \mathfrak{T}_{2\delta}$  and let  $L(\delta) < \infty$  be the Lipschitz constant of  $\mathfrak{l}|_{\mathfrak{T}_\delta}$  (recall  $\mathfrak{l}$  is concave). With theorem 4.1 we have

$$(3) \quad \left| \frac{1}{b^* - a^*} L^g(\gamma^*) - \mathfrak{l}(h) \right| \leq \frac{\overline{C}(\delta)}{b^* - a^*} + L(\delta) \|h - \rho(\gamma^*)\|$$

for any maximizer  $\gamma^*: [a^*, b^*] \rightarrow \overline{M}$  with  $\rho(\gamma^*) \in \mathfrak{T}_\delta$ . Choose  $\varepsilon = (\mathfrak{l}(h) - z)/10$  and consider  $T < \infty$  with

$$(4) \quad 2\overline{C}(\delta) + 2L(\delta)(2 \text{diam}(M, g_R) + 2 \text{std}(g_R)) \leq \varepsilon T.$$

Choose  $C > 0$  with  $1/C \leq \|h'\| \leq C$ , for all  $h' \in \mathfrak{T}^1$ . Increase  $T$ , if necessary, to be larger than  $CK(2 \text{diam}(M, g_R) + 2 \text{std}(g_R) + 1)/\delta$  (For the definition of  $K(\cdot)$  compare proposition 2.13).

For each  $i \in \{1, \dots, l\}$ , choose an infinite sequence of mutually disjoint intervals  $I_{ij} = [a_{ij}, b_{ij}]$ ,  $j \in \mathbb{N}$  such that  $b_{ij} - a_{ij}$  is an integral multiple of  $T$ ,  $b_{ij} - a_{ij} \rightarrow \infty$  and  $\mu_{ij} \xrightarrow{*} \mu_i$ , as  $j \rightarrow \infty$ , where  $\mu_{ij}$  denotes the probability measure evenly distributed along  $\gamma|_{I_{ij}}$ . Next consider the partition  $\{I_{ij\iota}\}_\iota$  of  $I_{ij}$  into intervals of length  $T$ . Obviously, the mean value of  $\{\rho(\gamma|_{I_{ij\iota}})\}_\iota$  is  $\rho(\gamma|_{I_{ij}})$ . Recall that we have  $\rho(\mu_{ij}) \xrightarrow{*} \rho(\mu_i)$ , as  $j \rightarrow \infty$ , and  $h$  is a convex combination of the  $\rho(\mu_i)$ . It is thus possible to choose a finite subcollection  $\{I_\kappa\}_{\kappa \in \{1, \dots, N\}}$  of the family  $\{I_{ij\iota}\}_{i,j,\iota}$  subject to two conditions. First, the mean value  $h'$  of the  $\rho(\gamma|_{J_\kappa})$  satisfies  $L(\delta)\|h' - h\| < \varepsilon/2$  and second the mean value of  $L^g(\gamma|_{J_\kappa})/T$  is smaller than  $z + \varepsilon$ . It represents no restriction on the generality to assume  $h' \in \mathfrak{T}_\delta$ , since this can always be achieved by increasing  $j$  and  $T$ . For later use note further that by raising  $T$  the stable norm of the  $\rho(\gamma|_{J_\kappa})$  can be assumed to lie between  $\frac{1}{2C}$  and  $2C$ . Let  $c_\kappa < d_\kappa$  denote the endpoints of  $J_\kappa$  and suppose that the intervals  $J_\kappa$  are indexed in increasing order, i.e.  $d_\kappa \leq c_{\kappa+1}$ . Let  $\overline{\gamma}: \mathbb{R} \rightarrow \overline{M}$  be any lift of  $\gamma$  to the Abelian cover. Choose deck transformations  $k_\kappa$  ( $0 \leq \kappa \leq N-1$ ,  $k_0 := \text{id}$ ) inductively such that

$$(5) \quad \left\| \sum_{\tau=1}^{\kappa} [\overline{\gamma}(d_\tau) + k_\tau - (\overline{\gamma}(c_\tau) + k_{\tau-1})] - \kappa T h' \right\| \leq \text{diam}(M, g_R) + \text{std}(g_R)$$

for all  $0 \leq \kappa \leq N-1$ . By the choice of  $T$  we know that

$$\text{dist}_{\|\cdot\|}(Th', \partial \mathfrak{T}) \geq T\delta \|h'\| \geq \frac{\delta}{C} T \geq K(2 \text{diam}(M, g_R) + 2 \text{std}(g_R) + 1).$$

This implies, using proposition 2.13, that for any pair of points  $(x, y) \in \overline{M} \times \overline{M}$  with  $y - x = Th'$ , the closed ball of radius  $2 \text{diam}(M, g_R) + 2 \text{std}(g_R)$  around  $y$  is contained in  $I^+(x)$ . Since we have, using (5),

$$(6) \quad \left\| \overline{\gamma}(d_\kappa) + k_\kappa - (\overline{\gamma}(c_\kappa) + k_{\kappa-1}) - Th' \right\| \leq 2 \text{diam}(M, g_R) + 2 \text{std}(g_R),$$

we obtain  $\bar{\gamma}(d_\kappa) + k_\kappa \in I^+(\bar{\gamma}(c_\kappa) + k_{\kappa-1})$  for all  $\kappa \leq N-1$ . From  $\bar{\gamma}(d_N) - \bar{\gamma}(c_1) = \rho^* + T \sum_{\tau=1}^N \rho(\gamma|_{J_\tau}) = \rho^* + TNh'$  for

$$\rho^* := \sum_{\tau=1}^{N-1} \bar{\gamma}(c_{\tau+1}) - \bar{\gamma}(d_\tau) = \sum_{\tau=1}^{N-1} \bar{\gamma}(c_{\tau+1}) + k_\tau - (\bar{\gamma}(d_\tau) + k_\tau)$$

and  $\bar{\gamma}(c_N) + k_{N-1} - \bar{\gamma}(c_1) = \rho^* + \sum_{\tau=1}^{N-1} \bar{\gamma}(d_\tau) + k_\tau - (\bar{\gamma}(c_\tau) + k_{\tau-1})$ , we obtain

$$(7) \quad \|[\bar{\gamma}(d_N) - (\bar{\gamma}(c_N) + k_{N-1})] - Th'\| \leq 2 \operatorname{diam}(M, g_R) + \operatorname{std}(g_R).$$

Thus with proposition 2.13 we get

$$\bar{\gamma}(d_N) \in I^+(\bar{\gamma}(c_N) + k_{N-1})$$

and we define  $k_N := \operatorname{id}$ .

With the deck transformations  $k_\kappa$  ( $0 \leq \kappa \leq N$ ) chosen, we can construct a new curve  $\tilde{\gamma}: \mathbb{R} \rightarrow \overline{M}$  as follows. Define

$$\tilde{\gamma}|_{(-\infty, c_1] \cup [d_N, \infty)} := \bar{\gamma}|_{(-\infty, c_1] \cup [d_N, \infty)}, \tilde{\gamma}|_{[d_\kappa, c_{\kappa+1}]} := \bar{\gamma}|_{[d_\kappa, c_{\kappa+1}]} + k_\kappa$$

and  $\tilde{\gamma}|_{[c_\kappa, d_\kappa]}$  a maximal geodesic joining  $\bar{\gamma}(c_\kappa) + k_{\kappa-1}$  with  $\bar{\gamma}(d_\kappa) + k_\kappa$ . Note that  $\tilde{\gamma}|_{[c_\kappa, d_\kappa]}$  is in general not parameterized by  $g_R$ -arclength. With the inequalities (3), (4), (6), (7) and  $L(\delta)\|h - h'\| < \varepsilon/2$  we conclude

$$|(d_\kappa - c_\kappa)^{-1} L^g(\tilde{\gamma}|_{[c_\kappa, d_\kappa]}) - \mathbf{l}(h)| < \varepsilon.$$

Consequently we have  $L^g(\tilde{\gamma}|_{[c_1, d_N]}) \geq \sum_{\kappa=1}^{N-1} L^g(\gamma|_{[d_\kappa, c_{\kappa+1}]}) + TN(\mathbf{l}(h) - \varepsilon)$ . But the assumptions imply that  $L^g(\gamma|_{[c_1, d_N]}) \leq \sum_{\kappa=1}^{N-1} L^g(\gamma|_{[d_\kappa, c_{\kappa+1}]}) + TN(z + \varepsilon)$ , since the mean value of the  $L^g(\gamma|_{J_\kappa})/T$  is smaller than  $z + \varepsilon$ . Hence we obtain  $L^g(\gamma|_{[c_1, d_N]}) < L^g(\tilde{\gamma}|_{[c_1, d_N]})$  and arrive at a contradiction to the maximization property of  $\gamma$ .  $\square$

So far proposition 6.1 does not give information whether the pregeodesics in the support of one of the ergodic measures of proposition 4.19 are lightlike or timelike. By the positivity of  $\mathbf{l}|_{\mathfrak{T}^\circ}$  we know that there has to be at least one invariant measure  $\mu$  with  $\operatorname{supp} \mu \cap \operatorname{Time}(M, [g]) \neq \emptyset$ .

A natural question arising at this point is: Does there exist an invariant measure which is supported entirely in  $\operatorname{Time}(M, [g])$  and if so, how many different ergodic measure of this kind are there necessarily? For maximizers this is equivalent to asking if there exists a sequence of tangents converging towards the light cones. In the geodesic parameterization of the timelike maximizers, this question is equivalent to asking whether the tangents are bounded in  $TM$ . Note that boundedness of the tangents is strictly stronger than completeness of the geodesics. An example of a complete maximal geodesic with unbounded tangents can be constructed from [18] theorem 8.1.

**Definition 6.4.** Let  $\alpha \in (\mathfrak{T}^*)^\circ$  and  $\tau: \overline{M} \rightarrow \mathbb{R}$  an calibration representing  $\alpha$ . A pregeodesic  $\gamma: \mathbb{R} \rightarrow M$  is calibrated by the calibration  $\tau$  if for one (hence every) lift  $\tilde{\gamma}: \mathbb{R} \rightarrow \overline{M}$  of  $\gamma$  and for all  $s < t \in \mathbb{R}$ , we have  $\tau(\tilde{\gamma}(t)) - \tau(\tilde{\gamma}(s)) = \mathbf{l}^*(\alpha) L^g(\gamma|_{[s, t]})$ .

For convenience of notation define for a calibration  $\tau: \overline{M} \rightarrow \mathbb{R}$  the set

$$\mathfrak{V}(\tau) := \{v \in T^{1,R}M \text{ future pointing} \mid \gamma_v \text{ is calibrated by } \tau\}.$$

The definition has the following immediate consequence.

**Corollary 6.5.** Let  $\tau: \overline{M} \rightarrow \mathbb{R}$  be a calibration representing  $\alpha \in (\mathfrak{T}^*)^\circ$ . Then the pregeodesic  $\gamma_v$  is a maximizer for any  $v \in \mathfrak{V}(\tau)$ .

**Proposition 6.6.** *Let  $(M, g)$  be a compact spacetime,  $\alpha \in (\mathfrak{T}^*)^\circ$  and  $\tau: \overline{M} \rightarrow \mathbb{R}$  a calibration representing  $\alpha$ . Further let  $\gamma: \mathbb{R} \rightarrow M$  be a future pointing maximizer calibrated by  $\tau$ . Then all limit measures of  $\gamma$  belong to  $\mathfrak{M}_\alpha$ . Moreover the image of the tangential mapping  $t \mapsto \dot{\gamma}(t)$  can be separated from  $\text{Light}(M, [g])$ , i.e. there exists  $\varepsilon = \varepsilon(\alpha) > 0$  such that  $\text{dist}(\dot{\gamma}(t), \text{Light}(M, [g])) \geq \varepsilon$  for all  $t \in \mathbb{R}$ .*

*Proof.* Let  $\mu$  be a probability limit measure of  $\gamma$  and  $[s_n, t_n] \subseteq \mathbb{R}$  such that

$$\frac{1}{t_n - s_n} \gamma_\#(\mathcal{L}^1|_{[s_n, t_n]}) \xrightarrow{*} \mu,$$

for  $n \rightarrow \infty$  (It poses no restriction to consider probability measures).

Choose an  $\alpha$ -equivariant smooth function  $\sigma: \overline{M} \rightarrow \mathbb{R}$  and an  $\alpha$ -invariant Lipschitz function  $\varphi: \overline{M} \rightarrow \mathbb{R}$  such that  $\tau = \sigma + \varphi$ . The differential of  $\sigma$  induces a smooth closed 1-form  $\omega_\sigma$  on  $M$ . Further  $\varphi$  induces a Lipschitz function  $\varphi'$  on  $M$ . Let  $\overline{\gamma}$  be a lift of  $\gamma$  to  $\overline{M}$ . We have

$$\begin{aligned} \frac{\tau(\overline{\gamma}(t_n)) - \tau(\overline{\gamma}(s_n))}{t_n - s_n} &= \frac{1}{t_n - s_n} \int_{s_n}^{t_n} \omega_\sigma(\dot{\gamma}(t)) dt + \frac{\varphi'(\gamma(t_n)) - \varphi'(\gamma(s_n))}{t_n - s_n} \\ &\rightarrow \int_{T^{1,R}M} \omega_\sigma d\mu = \alpha(\rho(\mu)) \end{aligned}$$

for  $n \rightarrow \infty$ . By assumption we have

$$\frac{1}{t_n - s_n} (\tau(\overline{\gamma}(t_n)) - \tau(\overline{\gamma}(s_n))) = \frac{\mathfrak{l}^*(\alpha)}{t_n - s_n} L^g(\gamma|_{[s_n, t_n]}) \rightarrow \mathfrak{l}^*(\alpha) \mathfrak{L}(\mu) \leq \mathfrak{l}^*(\alpha) \mathfrak{l}(\rho(\mu)).$$

Since  $\alpha(\rho(\mu)) \geq \mathfrak{l}^*(\alpha) \mathfrak{l}(\rho(\mu))$ , this implies equality, i.e.  $\alpha(\rho(\mu)) = \mathfrak{l}^*(\alpha) \mathfrak{L}(\mu) = \mathfrak{l}^*(\alpha) \mathfrak{l}(\rho(\mu))$  and consequently  $\mu \in \mathfrak{M}_\alpha$ .

By lemma 5.1 there exists  $\varepsilon > 0$  such that

$$\mathfrak{l}^*(\alpha) d(\overline{\gamma}(s), \overline{\gamma}(t)) = \tau(\overline{\gamma}(t)) - \tau(\overline{\gamma}(s)) \geq \varepsilon \text{dist}(\overline{\gamma}(s), \overline{\gamma}(t))$$

for all  $s \leq t$  and any lift  $\overline{\gamma}$  of  $\gamma$  to  $\overline{M}$ . Using the continuity of the pregeodesic flow and the fact that  $\text{Light}(M, [g])$  is  $\Phi$ -invariant, we see that the tangents of  $\gamma$  cannot approach  $\text{Light}(M, [g])$ .  $\square$

**Proposition 6.7.** *Let  $\alpha \in (\mathfrak{T}^*)^\circ$  and  $\tau: \overline{M} \rightarrow \mathbb{R}$  be a calibration representing  $\alpha$ . Then we have  $\text{supp } \mathfrak{M}_\alpha \subseteq \mathfrak{V}(\tau)$ , i.e. for any  $\mu \in \mathfrak{M}_\alpha$  and any  $v \in \text{supp } \mu$  the pregeodesic  $\gamma_v$  is calibrated by any calibration representing  $\alpha$ . The set  $\mathfrak{V}(\tau)$  is in particular not empty.*

*Proof.* Since  $\tau$  is  $\alpha$ -equivariant, the set  $\text{Def}(\partial\tau)$  and the function

$$\partial\tau: T^{1,R}\overline{M} \rightarrow \mathbb{R}, \overline{v} \mapsto \partial_{\overline{v}}\tau$$

are  $H_1(M, \mathbb{Z})_{\mathbb{R}}$ -invariant. Therefore we can define a bounded measurable function

$$\omega_\tau: T^{1,R}M \rightarrow \mathbb{R}, v \mapsto \partial_v\tau,$$

where  $\overline{v} \in T^{1,R}\overline{M}$  is any vector with  $\pi_*(\overline{v}) = v$ . Choose  $\sigma: \overline{M} \rightarrow \mathbb{R}$  and  $\varphi: \overline{M} \rightarrow \mathbb{R}$  as above.

Let  $\mu \in \mathfrak{M}_\alpha$ . By definition we have  $\alpha(\rho(\mu)) = \mathfrak{l}^*(\alpha) \mathfrak{L}(\mu)$ . Therefore we get, using lemma 4.13,

$$\mathfrak{l}^*(\alpha) \int |v|_g d\mu(v) = \alpha(\rho(\mu)) = \int_{T^{1,R}M} \omega_\sigma + \partial\varphi' d\mu = \int_{T^{1,R}M} \omega_\tau(v) d\mu(v).$$

Using Fubini's theorem and the  $\Phi$ -invariance of  $\mu$  we obtain, for all  $s < t \in \mathbb{R}$ ,

$$\begin{aligned} 0 &= \int_s^t \int_{T^{1,R}M} \omega_\tau(\Phi(v, t')) - \mathfrak{l}^*(\alpha)|\Phi(v, t')|_g d\mu(v) dt' \\ &= \int_{T^{1,R}M} \int_s^t \omega_\tau(\Phi(v, t')) - \mathfrak{l}^*(\alpha)|\Phi(v, t')|_g dt' d\mu(v) \\ &= \int_{T^{1,R}M} [\tau(\overline{\gamma}_v(t)) - \tau(\overline{\gamma}_v(s)) - \mathfrak{l}^*(\alpha)L^g(\gamma_v|_{[s,t]})] d\mu(v), \end{aligned}$$

where  $\overline{\gamma}_v$  is any lift of  $\gamma_v$  to  $\overline{M}$ . Note that the last equality follows since for any  $C^1$ -curve  $\gamma: I \rightarrow M$ , the map  $\tau \circ \gamma$  is differentiable almost everywhere and we can apply the fundamental theorem of calculus.

Since  $\tau$  is a calibration we have

$$\tau(\overline{\gamma}_v(t)) - \tau(\overline{\gamma}_v(s)) = \mathfrak{l}^*(\alpha)L^g(\gamma_v|_{[s,t]})$$

for  $\mu$ -almost all  $v \in T^{1,R}M$  and all  $s < t \in \mathbb{R}$ . Note that a set containing  $\mu$ -almost every point is dense in  $\text{supp } \mu$ . The general claim now follows from the continuity of  $\Phi$ .  $\square$

**Corollary 6.8.** *Let  $(M, g)$  be of class A. Then there exists a maximal ergodic measure  $\mu$  and  $\varepsilon > 0$  such that*

$$\text{dist}(\text{supp } \mu, \text{Light}(M, [g])) \geq \varepsilon.$$

*Proof.* Choose  $\alpha \in (\mathfrak{T}^*)^\circ$  and set  $\mathcal{K} := \{(h, t) \mid h \in \alpha^{-1}(1) \cap \mathfrak{T}, 0 \leq t \leq \mathfrak{l}(h)\}$ . Choose any  $h \in \alpha^{-1}(1) \cap \mathfrak{T}$  such that  $\alpha(h) = \mathfrak{l}^*(\alpha)\mathfrak{l}(h)$  (i.e.  $\alpha$  supports  $\mathfrak{l}$  at  $h$ ) and extremal points  $(h_i, t_i)$  ( $1 \leq i \leq b' \leq b$ ) of  $\mathcal{K}$  with  $(h, \mathfrak{l}(h)) \in \text{relint conv}\{(h_i, t_i)\}$ . Note that  $\mathfrak{l}(h) > 0$  since  $\alpha \in (\mathfrak{T}^*)^\circ$ . Then there exists  $1 \leq j \leq b'$  with  $t_j = \mathfrak{l}(h_j) > 0$  and it follows that  $\alpha(h_j) = \mathfrak{l}^*(\alpha)\mathfrak{l}(h_j)$ , since  $(h, \mathfrak{l}(h)) \in \text{relint conv}\{(h_i, t_i)\}$ .

Like in the proof of proposition 4.19 there exists a maximal ergodic measure  $\mu$  with  $\rho(\mu) \in \text{pos}\{h_j\}$ . Then we have  $\mu \in \mathfrak{M}_\alpha$ . By proposition 6.7 any  $\gamma$  with  $\gamma' \subseteq \text{supp } \mu$  is calibrated by any calibration representing  $\alpha$ . The claim now follows immediately with proposition 6.6.  $\square$

## 7. THE GRAPH THEOREM

**Theorem 7.1.** *Let  $(M, g)$  be of class A. Then the projection  $\pi_{TM}$  restricted to  $\text{supp } \mathfrak{M}_\alpha$  is injective for every  $\alpha \in \mathfrak{T}^*$ . Moreover there exists  $K = K(\alpha) < \infty$  such that the inverse of  $\pi_{TM}|_{\text{supp } \mathfrak{M}_\alpha}$  is  $1/2$ -Hölder-continuous on  $\pi_{TM}(\text{supp } \mathfrak{M}_\alpha)$  with constant  $K$ , i.e. we have*

$$\text{dist}(\pi_{TM}^{-1}(x), \pi_{TM}^{-1}(y))^2 \leq K \text{dist}(x, y)$$

for any  $x, y \in \pi_{TM}(\text{supp } \mathfrak{M}_\alpha)$ .

**Lemma 7.2.** *Let  $(M, g)$  be a compact spacetime. Then there exist  $\varepsilon, \delta, \eta > 0$  and  $K < \infty$  such that for all geodesically convex neighborhoods  $U$  in  $(M, g)$  and all future pointing pregeodesics  $x_1, x_2: [-\varepsilon, \varepsilon] \rightarrow U$  with*

$$\text{dist}(x_1(0), x_2(0)) \leq \delta \text{ and } \text{dist}(x'_1(0), x'_2(0))^2 \geq K \text{dist}(x_1(0), x_2(0)),$$

*there exist future pointing  $C^1$ -curves  $y_1, y_2: [-\varepsilon, \varepsilon] \rightarrow U$  with  $y_1(-\varepsilon) = x_1(-\varepsilon)$ ,  $y_1(\varepsilon) = x_2(\varepsilon)$ ,  $y_2(-\varepsilon) = x_2(-\varepsilon)$ ,  $y_2(\varepsilon) = x_1(\varepsilon)$  and*

$$L^g(y_1) + L^g(y_2) - L^g(x_1) - L^g(x_2) \geq \eta \text{dist}(\dot{x}_1(0), \dot{x}_2(0))^2.$$

**Remark 7.3.** *The formulation of lemma 7.2 is optimal. Counterexamples can be easily constructed in any Minkowski space of dimension at least 3.*

Theorem 7.1 follows in exactly the same way from lemma 7.2, theorem 2 in [11] follows from the lemma therein.

The content of the following lemma are technical steps in the proof of lemma 7.2.

**Lemma 7.4.** *Let  $(M, g)$  be a compact spacetime and  $g_R$  a Riemannian metric on  $M$ .*

(i) *Denote by  $\angle(v, w)$  the angle relative to  $g_R$  between  $v$  and  $w \in TM_p$ . Then there exists  $\tilde{\varepsilon} = \tilde{\varepsilon}(g, g_R) > 0$  such that*

$$(8) \quad -g(v, w) - |v|_g |w|_g \geq \tilde{\varepsilon} |v| |w| \sin^2 \angle(v, w),$$

*for any pair of future pointing vectors  $v, w \in TM$  with  $\pi_{TM}(v) = \pi_{TM}(w)$ .*

(ii) *There exists  $\tilde{C} = \tilde{C}(g, g_R) < \infty$  with*

$$|g(v, v)| \leq \tilde{C} |v| \text{dist}(v, \text{Light}(M, [g]))$$

*for all future pointing  $v \in TM$ .*

*Proof.* (i) Note that (8) is positively homogenous of degree 2. Therefore it suffices to verify (8) for  $g_R$ -unit vectors. For  $g_R$ -unit vectors we have

$$\sin^2 \angle(v, w) = 1 - g_R(v, w)^2 = (1 + g_R(v, w)) \frac{1}{2} |v - w|^2 \leq |v - w|^2.$$

W.l.o.g. we can assume that there exists a timelike future pointing  $g$ -unit vector field  $X \in \Gamma(TM)$  such that  $g_R = g + 2g(X, \cdot) \otimes g(X, \cdot)$ . The general case follows from this special case, since any two Riemannian metrics on a compact manifold are equivalent.

Under the assumption that  $g_R = g + 2g(X, \cdot) \otimes g(X, \cdot)$ , it is a simple calculation to show

$$(9) \quad -g(v, w) - |v|_g |w|_g \geq \frac{1}{2} |v - w|^2$$

for all future pointing  $g_R$ -unit vectors  $v, w \in TM$  with  $\pi_{TM}(v) = \pi_{TM}(w)$ . Denote with  $v_0 := -g(v, X) = g_R(v, X)$  and  $\bar{v} := v - v_0 X$ . Then we have  $\frac{1}{2} |v - w|^2 = 1 - g_R(v, w)$  and  $-g(v, w) = v_0 w_0 - g_R(\bar{v}, \bar{w})$ . Note that  $|v|_g = \sqrt{v_0^2 - |\bar{v}|^2} = \sqrt{2v_0^2 - 1}$ , since by our choice of  $g_R$ ,  $1 = |v|^2 = v_0^2 + |\bar{v}|^2$ . Then (9) is equivalent to

$$(2v_0 w_0 - 1)^2 \geq (2v_0^2 - 1)(2w_0^2 - 1),$$

which is always satisfied.

(ii) Consider  $w \in \text{Light}(M, [g])$  with  $|v - w| = \text{dist}(v, \text{Light}(M, [g]))$ . Note that we have  $|w| \leq 2|v|$  and therefore

$$\begin{aligned} -g(v, v) &= -g(v, v) + g(w, w) = -2 \int_0^1 g((1-t)w + tv, v - w) dt \\ &\leq 2\Lambda_{g, g_R} \sup_{t \in [0, 1]} |(1-t)w + tv| |v - w| \leq 4\Lambda_{g, g_R} |v| |v - w| \\ &=: \tilde{C} |v| \text{dist}(v, \text{Light}(M, [g])). \end{aligned}$$

□

*Proof of Lemma 7.2.* To keep the exposition clear and simple we assume that the Riemannian opening angles of the time cones  $\text{Time}(M, [g])_p$  are bounded from above by  $\pi/2$  for all  $p \in M$ . This poses no restriction since a different choice of Riemannian metric alters only the numerical values of the constants.

Choose a finite cover  $\{U_i\}_{1 \leq i \leq N}$  of  $(M, g)$  by geodesically convex neighborhoods such that  $(\bar{U}_i, g|_{\bar{U}_i})$  is globally hyperbolic and there exists  $L_0 < \infty$  such that  $\exp_r^{-1}|_{U_i}$  is  $L_0$ -bi-Lipschitz for all  $1 \leq i \leq N$  and all  $r \in U_i$ . Further choose  $\varepsilon \in (0, 1)$  such that  $4\varepsilon$  is a Lebesgue number of  $\{U_i\}_{1 \leq i \leq N}$  and  $\delta \leq \varepsilon \min\{\frac{1}{4}, \frac{1}{2C_{g, g_R}}\}$ .



Now for any convex normal neighborhood  $U$  in  $(M, g)$  and every pair of pregeodesics  $x_{1,2}: [-\varepsilon, \varepsilon] \rightarrow U$  with  $\text{dist}(x_1(0), x_2(0)) \leq \delta$  there exists  $i \in \{1, \dots, N\}$  with  $x_1, x_2 \subseteq U_i$ . Therefore it suffices to consider the case  $U = U_i$  for some  $i \in \{1, \dots, N\}$ .

Set  $v := x'_1(t)$ ,  $w := x'_2(t)$ ,  $p := x_1(0)$  and  $q := x_2(0)$ . Then the conditions of the lemma reformulate to  $\text{dist}^2(v, w) \geq K \text{dist}(p, q)$  and  $\text{dist}(p, q) \leq \delta$ .

We will show that there exists a future pointing curve  $y_1: [-\varepsilon, \varepsilon] \rightarrow M$  with  $y_1(-\varepsilon) = x_1(-\varepsilon)$ ,  $y_1(\varepsilon) = x_2(\varepsilon)$  and

$$L^g(y_1) - L^g(x_1|_{[-\varepsilon, 0]}) - L^g(x_2|_{[0, \varepsilon]}) \geq \frac{\eta}{2} \text{dist}^2(v, w).$$

The construction of  $y_2$  follows by exchanging  $x_1$  and  $x_2$ .

(i): The first step will be to show that  $x_2(\varepsilon) \in I^+(x_1(-\varepsilon))$  and

$$\text{dist}(\exp_{x_1(-\varepsilon)}^{-1}(x_2(\varepsilon)), \text{Light}(M, [g])) \geq \varepsilon_0 \text{dist}^2(v, w),$$

where  $\varepsilon_0 > 0$  depends only on  $g$  and  $g_R$ . Denote with  $w_1$  the  $g$ -parallel transport of  $w$  along the unique geodesic in  $U$  between  $q$  and  $p$  and with  $w_2$  the  $g_R$ -normalization of  $w_1$ . Note that  $w_1$  and  $w_2$  are future pointing. Since  $g_R$  is continuous and the parallel transport is the solution of an ordinary differential equation depending only on  $g$ , we have

$$\text{dist}(w_2, w) \leq C_1 \text{dist}(p, q)$$

for some  $C_1 < \infty$  depending only on  $g$  and  $g_R$ . Set  $\chi_2(t) := \pi_{TM}(\Phi(w_2, t))$ . Note that we have  $\chi_2([0, \varepsilon]) \subseteq U$  by our choice of  $\varepsilon > 0$ .

Consider the function  $\mathcal{D}: U \rightarrow \mathbb{R}$ ,  $r \mapsto d_U(x_1(-\varepsilon), r)$ , where  $d_U$  denotes the time separation of  $(U, g|_U)$ . Recall that  $\mathcal{D}$  is smooth at  $r \in I_U^+(x_1(-\varepsilon))$  with past pointing timelike  $g$ -gradient  $\nabla \mathcal{D}(r)$ . We have  $\chi_2(s) \in I_U^+(x_1(-\varepsilon))$  if  $\text{dist}(v, w) > 0$  and  $s \in (0, \varepsilon]$ . We can assume  $\text{dist}(v, w) > 0$ , since there is nothing to prove for  $\text{dist}(v, w) = 0$ .

For  $s \in (0, \varepsilon)$  we can apply lemma 7.4 (i) and obtain (recall that  $|\chi'_2| \equiv 1$ )

$$\begin{aligned} \frac{1}{2}(\mathcal{D}^2(\chi_2(\varepsilon)) - \mathcal{D}^2(\chi_2(s))) &= \int_s^\varepsilon -g(-\mathcal{D}(\chi_2(\sigma))\nabla \mathcal{D}(\chi_2(\sigma)), \chi'_2(\sigma))d\sigma \\ &\geq \int_s^\varepsilon |-\mathcal{D}\nabla \mathcal{D}|_g |\chi'_2(\sigma)|_g d\sigma + \tilde{\varepsilon} \int_s^\varepsilon \sin^2 \angle(-\mathcal{D}\nabla \mathcal{D}, \chi'_2(\sigma)) |-\mathcal{D}\nabla \mathcal{D}| d\sigma \\ &\geq \tilde{\varepsilon} \int_s^\varepsilon \sin^2 \angle(-\mathcal{D}\nabla \mathcal{D}, \chi'_2(\sigma)) |-\mathcal{D}\nabla \mathcal{D}| d\sigma. \end{aligned}$$

For  $r \in U$  denote with  $\zeta: [0, t_r] \rightarrow U$  the unique pregeodesic connecting  $x_1(-\varepsilon)$  with  $r$ . Define the vector field  $X \in \Gamma^\infty(T^{1,R}U)$  by  $X_r := \zeta'(t_r)$ . Then we can choose a constant  $\Lambda = \Lambda(g, g_R, \varepsilon) < \infty$  such that  $X$  is  $\Lambda$ -Lipschitz at all  $r \in U$  with  $\text{dist}(r, x_1(-\varepsilon)) \geq \varepsilon/C_{g, g_R}$ . Since  $(M, g)$  is of class A we have, using corollary 2.12,

$$(10) \quad \varepsilon + \sigma = L^{g_R}(x_1|_{[-\varepsilon, 0]} * \chi_2|_{[0, \sigma]}) \leq C_{g, g_R} \text{dist}(x_1(-\varepsilon), \chi_2(\sigma))$$

for all  $\sigma \in [0, \varepsilon]$ . We claim that there exists  $\varepsilon_1 = \varepsilon_1(g, g_R, \varepsilon) > 0$  such that

$$(11) \quad \varepsilon_1 \sin \angle(\chi'_2(s), X_{\chi_2(s)}) \leq \sin \angle(\chi'_2(\sigma), X_{\chi_2(\sigma)})$$

for all  $s \in [0, \varepsilon]$  and all  $\sigma \in [s, \varepsilon]$ . Abbreviate  $\zeta_\sigma := \zeta_{\chi_2(\sigma)}$  and  $t_\sigma := t_{\chi_2(\sigma)}$ . Let  $L = L(g, g_R, \varepsilon) < \infty$  be a Lipschitz constant of  $\Phi|_{T^{1,R}M \times [-\varepsilon, \varepsilon]}$ . Then we have

$$\begin{aligned} \text{dist}(\chi'_2(s), X_{\zeta_\sigma(t_\sigma - (\sigma - s))}) &= \text{dist}(\chi'_2(s), \zeta'_\sigma(t_\sigma - (\sigma - s))) \\ &\leq L \text{dist}(\chi'_2(\sigma), \zeta'_\sigma(t_\sigma)) = L \text{dist}(\chi'_2(\sigma), X_{\chi_2(\sigma)}) \end{aligned}$$

and with the  $\Lambda$ -Lipschitz continuity of  $X$  we get

$$\begin{aligned} \text{dist}(X_{\zeta_\sigma(t_\sigma - (\sigma - s))}, X_{\chi_2(s)}) \\ \leq \Lambda \text{dist}(\zeta_\sigma(t_\sigma - (\sigma - s)), \chi_2(s)) \leq \Lambda \text{dist}(\zeta'_\sigma(t_\sigma - (\sigma - s)), \chi'_2(s)) \\ \leq \Lambda L \text{dist}(\zeta'_\sigma(t_\sigma), \chi'_2(\sigma)) = \Lambda L \text{dist}(X_{\chi_2(\sigma)}, \chi'_2(\sigma)). \end{aligned}$$

Summing up we get

$$\begin{aligned} \text{dist}(\chi'_2(s), X_{\chi_2(s)}) &\leq \text{dist}(\chi'_2(s), X_{\zeta_\sigma(t_\sigma - (\sigma - s))}) + \text{dist}(X_{\zeta_\sigma(t_\sigma - (\sigma - s))}, X_{\chi_2(s)}) \\ &\leq (1 + \Lambda)L \text{dist}(\chi'_2(\sigma), X_{\chi_2(\sigma)}). \end{aligned}$$

By our choice of Riemannian metric on  $TM$  we have  $\text{dist}(\chi'_2(\sigma), X_{\chi_2(\sigma)}) = |\chi'_2(\sigma) - X_{\chi_2(\sigma)}|$  for all  $\sigma \in [0, \varepsilon]$ . It is an elementary fact that

$$\sin \angle(\chi'_2(\sigma), X_{\chi_2(\sigma)}) \leq |(\chi'_2(\sigma) - X_{\chi_2(\sigma)})| \leq \sqrt{2} \sin \angle(\chi'_2(\sigma), X_{\chi_2(\sigma)})$$

for all  $\sigma \in [0, \varepsilon]$ , since by our assumption  $\angle(\chi'_2(\sigma), X_{\chi_2(\sigma)}) \leq \pi/2$ . Combining the last two inequalities we obtain (11) for  $\varepsilon_1 := \frac{1}{(1+\Lambda)L\sqrt{2}}$ .

Note that

$$X_r = \frac{-\mathcal{D}(r)\nabla\mathcal{D}_r}{|-\mathcal{D}(r)\nabla\mathcal{D}_r|}$$

for  $r \in I_U^+(x_1(-\varepsilon))$ . Therefore we get

$$\begin{aligned} \int_s^\varepsilon \sin^2 \angle(-\mathcal{D}\nabla\mathcal{D}, \chi'_2(\sigma)) |\mathcal{D}\nabla\mathcal{D}| d\sigma \\ \geq \varepsilon_1 \sin^2 \angle(-\mathcal{D}\nabla\mathcal{D}, \chi'_2(s)) \int_s^\varepsilon |\mathcal{D}\nabla\mathcal{D}|(\chi_2(\sigma)) d\sigma. \end{aligned}$$

Recall that

$$(\exp_{x_1(-\varepsilon)}^{-1})_* r(\mathcal{D}(r)\nabla\mathcal{D}(r)) = \exp_{x_1(-\varepsilon)}^{-1}(r)$$

for all  $r \in I_U^+(x_1(-\varepsilon))$ . Using the bi-Lipschitz continuity of  $\exp_{x_1(-\varepsilon)}^{-1}|_U$  we see that

$$|\mathcal{D}\nabla\mathcal{D}|(\chi_2(\sigma)) \geq \frac{1}{L_0} |\exp_{x_1(-\varepsilon)}^{-1}(\chi_2(\sigma))|$$

and

$$|\exp_{x_1(-\varepsilon)}^{-1}(\chi_2(\sigma))| \geq \frac{1}{2C_{g,g_R}L_0^2} |\exp_{x_1(-\varepsilon)}^{-1}(\chi_2(\varepsilon))|$$

for all  $\sigma \in [0, \varepsilon]$ . Then we have

$$\begin{aligned} \int_s^\varepsilon \mathcal{D}(\chi_2(\sigma)) |\nabla\mathcal{D}(\chi_2(\sigma))| d\sigma &\geq \frac{1}{L_0} \int_s^\varepsilon |\exp_{x_1(-\varepsilon)}^{-1}(\chi_2(\sigma))| d\sigma \\ &\geq \frac{\varepsilon - s}{2C_{g,g_R}L_0^3} |\exp_{x_1(-\varepsilon)}^{-1}(\chi_2(\varepsilon))| \\ &=: \varepsilon_2(\varepsilon - s) |\exp_{x_1(-\varepsilon)}^{-1}(\chi_2(\varepsilon))|, \end{aligned}$$

using 10. Note that  $-\mathcal{D}(\chi_2(s))\nabla\mathcal{D}(\chi_2(s)) \rightarrow v$  for  $s \downarrow 0$ . Consequently we get

$$\sin^2 \angle(-\mathcal{D}\nabla\mathcal{D}(\chi_2(s)), \chi'_2(s)) \rightarrow \sin^2 \angle(v, w_2)$$

for  $s \downarrow 0$ .

Recall that  $\text{dist}(w, w_2) \leq C_1 \text{dist}(p, q)$ . We have

$$\begin{aligned} \sqrt{2} \sin \angle(x'_1(0), \chi'_2(0)) &= \sqrt{2} \sin \angle(v, w_2) \geq |v - w_2| = \text{dist}(v, w_2) \\ &\geq \text{dist}(v, w) - \text{dist}(w, w_2) \geq \text{dist}(v, w) - C_1 \text{dist}(p, q) \\ &\geq \text{dist}(v, w) - \frac{C_1}{K} \text{dist}^2(v, w) \geq \frac{1}{2} \text{dist}(v, w), \end{aligned}$$

for  $K \geq 2C_1(\sqrt{2} + 1)$  (Recall that by our choice of Riemannian metric on  $TM$ ,  $\text{dist}(v, w) \leq \text{dist}(p, q) + \sqrt{2} \leq \delta + \sqrt{2} \leq 1 + \sqrt{2}$ ).

Combining the deductions above we get

$$\begin{aligned}
 |\exp_{x_1(-\varepsilon)}^{-1}(\chi_2(\varepsilon))|_g^2 &= d_U^2(x_1(-\varepsilon), \chi_2(\varepsilon)) = \mathcal{D}^2(\chi_2(\varepsilon)) \\
 &\geq \mathcal{D}^2(\chi_2(0)) + \tilde{\varepsilon}\varepsilon_1\varepsilon_2\varepsilon \sin^2 \angle(v, w_2) |\exp_{x_1(-\varepsilon)}^{-1}(\chi_2(\varepsilon))| \\
 &\geq \mathcal{D}^2(\chi_2(0)) + \frac{\tilde{\varepsilon}\varepsilon_1\varepsilon_2\varepsilon}{8} \text{dist}^2(v, w) |\exp_{x_1(-\varepsilon)}^{-1}(\chi_2(\varepsilon))| \\
 &\geq \frac{\tilde{\varepsilon}\varepsilon_1\varepsilon_2\varepsilon}{8} \text{dist}^2(v, w) |\exp_{x_1(-\varepsilon)}^{-1}(\chi_2(\varepsilon))| \\
 &=: \varepsilon_3 \text{dist}^2(v, w) |\exp_{x_1(-\varepsilon)}^{-1}(\chi_2(\varepsilon))|.
 \end{aligned}$$

With lemma 7.4 (ii) follows

$$\frac{\varepsilon_3}{C} \text{dist}^2(v, w) \leq \text{dist}(\exp_{x_1(-\varepsilon)}^{-1}(\chi_2(\varepsilon)), \text{Light}(M, [g])_{x_1(-\varepsilon)}).$$

From  $\text{dist}(w, w_2) \leq C_1 \text{dist}(p, q)$  we get  $\text{dist}(x'_2(\sigma), \chi'_2(\sigma)) \leq LC_1 \text{dist}(p, q)$  for all  $\sigma \in [0, \varepsilon]$  and therefore  $\text{dist}(x_2(\varepsilon), \chi_2(\varepsilon)) \leq LC_1 \text{dist}(p, q)$ . Using the bi-Lipschitz continuity of  $\exp_{x_1(-\varepsilon)}^{-1}|_U$  yields

$$|\exp_{x_1(-\varepsilon)}^{-1}(x_2(\varepsilon)) - \exp_{x_1(-\varepsilon)}^{-1}(\chi_2(\varepsilon))| \leq L_0 LC_1 \text{dist}(p, q).$$

For  $K \geq \frac{2\tilde{C}L_0LC_1}{\varepsilon_3}$  (increase  $K$  if necessary) and  $\varepsilon_0 := \frac{\varepsilon_3}{2C}$  we get

$$\exp_{x_1(-\varepsilon)}^{-1}(x_2(\varepsilon)) \in \text{Time}(M, [g])_{x_1(-\varepsilon)}$$

and

$$\text{dist}(\exp_{x_1(-\varepsilon)}^{-1}(x_2(\varepsilon)), \text{Light}(M, [g])_{x_1(-\varepsilon)}) \geq \varepsilon_0 \text{dist}^2(v, w).$$

(ii) Now we will show that there exists  $\eta > 0$  such that

$$\mathcal{D}(x_2(\varepsilon)) - L^g(x_1|_{[-\varepsilon, 0]}) - L^g(x_2|_{[0, \varepsilon]}) \geq \eta \text{dist}^2(v, w)$$

for  $\text{dist}^2(v, w) \geq K \text{dist}(p, q)$  and  $\text{dist}(p, q) \leq \delta$  for  $\delta$  as above and  $K$  sufficiently large.

Denote with  $\chi_3: [0, \bar{\varepsilon}] \rightarrow U$  the unique pregeodesic connecting  $x_1(0)$  with  $x_2(\varepsilon)$ . By part (i)  $\chi_3$  is future pointing timelike. Using the same arguments as in step (i) we obtain, for all  $s \in (0, \varepsilon)$ ,

$$\begin{aligned}
 \mathcal{D}(\chi_3(\bar{\varepsilon})) - \mathcal{D}(\chi_3(s)) &= \int_s^{\bar{\varepsilon}} g(\nabla \mathcal{D}(\chi_3(\sigma)), \dot{\chi}_3(\sigma)) d\sigma \\
 &\geq \int_s^{\bar{\varepsilon}} |\nabla \mathcal{D}|_g |\chi'_3(\sigma)|_g d\sigma + \tilde{\varepsilon} \int_s^{\bar{\varepsilon}} \sin^2 \angle(-\nabla \mathcal{D}, \chi'_3(\sigma)) |-\nabla \mathcal{D}| d\sigma \\
 &= L^g(\chi_3|_{[s, \bar{\varepsilon}]}) + \tilde{\varepsilon} \int_s^{\bar{\varepsilon}} \sin^2 \angle(-\nabla \mathcal{D}, \chi'_3(\sigma)) |-\nabla \mathcal{D}| d\sigma \\
 &\geq L^g(\chi_3|_{[s, \bar{\varepsilon}]}) + \tilde{\varepsilon}\varepsilon_1 \sin^2 \angle(-\nabla \mathcal{D}, \chi'_3(s)) \int_s^{\bar{\varepsilon}} |-\nabla \mathcal{D}|(\chi_3(\sigma)) d\sigma \\
 &\geq L^g(\chi_3|_{[s, \bar{\varepsilon}]}) + \tilde{\varepsilon}\varepsilon_1 \sin^2 \angle(-\nabla \mathcal{D}, \chi'_3(s)) \frac{\bar{\varepsilon} - s}{\sqrt{\Lambda_{g, g_R}}}.
 \end{aligned}$$

The last inequality follows from our choice of  $\Lambda_{g, g_R} < \infty$ , i.e.

$$1 = |g(-\nabla \mathcal{D}, -\nabla \mathcal{D})| \leq \Lambda_{g, g_R} g_R(-\nabla \mathcal{D}, -\nabla \mathcal{D}).$$

Further we have

$$\begin{aligned} \sin^2 \angle(-\nabla \mathcal{D}(\chi_3(s)), \chi'_3(s)) \\ = \sin^2 \angle(-\mathcal{D}(\chi_3(s)) \nabla \mathcal{D}(\chi_3(s)), \chi'_3(s)) \rightarrow \sin^2 \angle(v, \chi'_3(0)) \end{aligned}$$

for  $s \downarrow 0$ .

From  $\text{dist}(\chi_2(\varepsilon), x_2(\varepsilon)) \leq LC_1 \text{dist}(p, q)$  (see step (i)) we get  $\text{dist}(\chi'_3(0), \chi'_2(0)) \leq C_2 \text{dist}(p, q)$  and

$$|L^g(x_2|_{[0, \varepsilon]}) - L^g(\chi_3|_{[0, \bar{\varepsilon}]})| \leq C_3 \text{dist}(p, q) \varepsilon$$

for some  $C_2, C_3 < \infty$  depending only on  $g$  and  $g_R$ . Like in step (i) we get

$$\begin{aligned} \sqrt{2} \sin \angle(v, \chi'_3(0)) &\geq |v - \chi'_3(0)| = \text{dist}(v, \chi'_3(0)) \\ &\geq \text{dist}(v, w) - \text{dist}(w, \chi'_3(0)) \geq \text{dist}(v, w) - C_2 \text{dist}(p, q) \\ &\geq \text{dist}(v, w) - \frac{C_2}{K} \text{dist}^2(v, w) \geq \frac{1}{2} \text{dist}(v, w), \end{aligned}$$

for  $K \geq 2C_2(\sqrt{2} + 1)$  (Increase  $K$  if necessary). Note that

$$\bar{\varepsilon} \geq \text{dist}(x_1(0), x_2(\varepsilon)) \geq \text{dist}(x_2(\varepsilon), x_2(0)) - \text{dist}(x_2(0), x_1(0)) \geq \frac{\varepsilon}{C_{g, g_R}} - \delta \geq \frac{\varepsilon}{2C_{g, g_R}},$$

by our choice of  $\delta > 0$ . By construction we have  $L^g(x_1|_{[-\varepsilon, 0]}) = \mathcal{D}(\chi_3(0))$  and

$$\begin{aligned} d_U(x_1(-\varepsilon), x_2(\varepsilon)) - L^g(x_1|_{[-\varepsilon, 0]}) - L^g(x_2|_{[0, \varepsilon]}) \\ \geq d_U(x_1(-\varepsilon), x_2(\varepsilon)) - L^g(x_1|_{[-\varepsilon, 0]}) - L^g(\chi_3|_{[0, \bar{\varepsilon}]}) - C_3 \text{dist}(p, q) \varepsilon \\ \geq \frac{\tilde{\varepsilon} \varepsilon_1 \bar{\varepsilon}}{\sqrt{\Lambda_{g, g_R}}} \sin^2 \angle(v, \chi'_3(0)) - C_3 \text{dist}(p, q) \varepsilon \\ \geq \frac{\tilde{\varepsilon} \varepsilon_1 \bar{\varepsilon}}{8\sqrt{\Lambda_{g, g_R}}} \text{dist}^2(v, w) - C_3 \varepsilon \text{dist}(p, q) \\ \geq \frac{\tilde{\varepsilon} \varepsilon_1 \varepsilon}{16\sqrt{\Lambda_{g, g_R}} C_{g, g_R}} \text{dist}^2(v, w) - C_3 \varepsilon \text{dist}(p, q) \\ \geq \frac{\tilde{\varepsilon} \varepsilon_1 \varepsilon}{32\sqrt{\Lambda_{g, g_R}} C_{g, g_R}} \text{dist}^2(v, w) \end{aligned}$$

for  $K \geq \frac{32\sqrt{\Lambda_{g, g_R}} C_3 C_{g, g_R}}{\tilde{\varepsilon} \varepsilon_1}$  (Increase  $K$  if necessary).

(iii) Exchanging the roles of  $x_1|_{[0, \varepsilon]}$  and  $x_2|_{[-\varepsilon, 0]}$  we get

$$\begin{aligned} d_U(x_1(-\varepsilon), x_2(\varepsilon)) + d_U(x_2(-\varepsilon), x_1(\varepsilon)) \\ - L^g(x_1|_{[-\varepsilon, \varepsilon]}) - L^g(x_2|_{[-\varepsilon, \varepsilon]}) \geq \eta \text{dist}(v, w)^2 \end{aligned}$$

for  $\eta \leq \frac{\tilde{\varepsilon} \varepsilon_1 \varepsilon}{4\sqrt{\Lambda_{g, g_R}}}$ . Choose for  $y_1, y_2$  monotone reparameterizations of the maximal pregeodesics connecting  $x_1(-\varepsilon)$  with  $x_2(\varepsilon)$  resp.  $x_2(-\varepsilon)$  with  $x_1(\varepsilon)$ . We have seen that  $y_1$  and  $y_2$  are timelike with  $L^g(y_1) = d_U(x_1(-\varepsilon), x_2(\varepsilon))$  as well as  $L^g(y_2) = d_U(x_2(-\varepsilon), x_1(\varepsilon))$ . □

After treating the general case, we turn our attention toward the intersection  $\text{supp } \mu \cap \text{Time}(M, [g])$ . We have seen in corollary 6.8 that there exists at least one maximal measure  $\mu$  with  $\text{supp } \mu \subseteq \text{Time}(M, [g])$ . Therefore the set of tangent vectors addressed in this special case is not empty. We will recover the Lipschitz continuity of  $(\pi_{TM}|_{\text{supp } \mathfrak{M}_\alpha})^{-1}$  for at  $v \in \text{supp } \mu \cap \text{Time}(M, [g])$  with  $\mu$  maximal, which is well known in the Riemannian case.

**Proposition 7.5.** *Let  $(M, g)$  be of class A. Then for every  $\alpha \in \mathfrak{T}^*$  and every  $\kappa > 0$  there exists  $K' = K'(\alpha, \kappa) < \infty$  such that for every  $v \in \text{supp } \mathfrak{M}_\alpha \cap \text{Time}(M, [g])^\kappa$  the inverse of  $\pi_{TM}|_{\text{supp } \mathfrak{M}_\alpha \cap \text{Time}(M, [g])^\kappa}$  is Lipschitz at  $\pi(v)$  with Lipschitz constant  $K'$ , i.e.*

$$\text{dist}(v, \pi^{-1}(y)) \leq K' \text{dist}(\pi(v), y)$$

for any  $y \in \pi_{TM}(\text{supp } \mathfrak{M}_\alpha)$ .

We obtain the following immediate corollary.

**Theorem 7.6.** *Let  $(M, g)$  be of class A. Then for every  $\alpha \in \mathfrak{T}^*$  and every  $\kappa > 0$  the inverse of  $\pi: \text{supp } \mathfrak{M}_\alpha \cap \text{Time}(M, [g])^\kappa \rightarrow M$  is Lipschitz with constant  $K' = K'(\alpha, \kappa) < \infty$ , i.e.*

$$\text{dist}(\pi^{-1}(x), \pi^{-1}(y)) \leq K' \text{dist}(x, y)$$

for any  $x, y \in \pi_{TM}(\mathfrak{M}_\alpha \cap \text{Time}(M, [g])^\kappa)$ .

We can strengthen the claim for  $\alpha \in (\mathfrak{T}^*)^\circ$ . With proposition 6.7 we know that any pregeodesic in  $\text{supp } \mathfrak{M}_\alpha$  is calibrated by every calibration representing  $\alpha$ . Since every calibrated pregeodesic  $\gamma$  is timelike and satisfies  $\gamma'(t) \in \text{Time}(M, [g])^\kappa$  for some  $\kappa = \kappa(\alpha) > 0$  and every  $t \in \mathbb{R}$ , we can drop the condition “ $v \in \text{Time}(M, [g])^\kappa$ ” for  $v \in \text{supp } \mathfrak{M}_\alpha$  in theorem 7.6. Further we can extend the result to all curves calibrated by a calibration representing  $\alpha$ .

**Theorem 7.7.** *Let  $(M, g)$  be of class A. Then for all  $\alpha \in (\mathfrak{T}^*)^\circ$  the restriction  $\pi_{TM}|_{\mathfrak{V}(\tau)}$  is injective and there exists  $K'' = K''(\alpha) < \infty$  such that the inverse of  $\pi_{TM}|_{\mathfrak{V}(\tau)}$  is  $K''$ -Lipschitz for all calibrations  $\tau$  representing  $\alpha$ .*

**Lemma 7.8.** *Let  $\kappa' > 0$ . Then there exist  $\varepsilon, \delta, \eta > 0$  and  $K' < \infty$  such that for every pair of future pointing pregeodesics  $x_1, x_2: [-\varepsilon, \varepsilon] \rightarrow M$  with*

- (i)  $\text{dist}(x_1(0), x_2(0)) \leq \delta$ ,
- (ii)  $\text{dist}(x'_1(0), x'_2(0)) \geq K' \text{dist}(x_1(0), x_2(0))$  and
- (iii)  $x'_1(0)$  and  $x'_2(0) \in \text{Time}(M, [g])^{\kappa'}$ ,

*there exist future pointing  $C^1$ -curves  $y_1, y_2: [-\varepsilon, \varepsilon] \rightarrow M$  with  $y_1(-\varepsilon) = x_1(-\varepsilon)$ ,  $y_1(\varepsilon) = x_2(\varepsilon)$ ,  $y_2(-\varepsilon) = x_2(-\varepsilon)$ ,  $y_2(\varepsilon) = x_1(\varepsilon)$  and*

$$L^g(y_1) + L^g(y_2) - L^g(x_1) - L^g(x_2) \geq \eta \text{dist}(\dot{x}_1(0), \dot{x}_2(0))^2.$$

*Proof of proposition 7.5.* With theorem 7.1 we know that  $\pi_{TM}|_{\text{supp } \mathfrak{M}_\alpha}$  is injective and the inverse is 1/2-Hölder continuous. Therefore we can assume that for  $v, w \in \text{supp } \mathfrak{M}_\alpha$  sufficiently close with (w.l.o.g.)  $v \in \text{Time}(M, [g])^\kappa$  we have  $w \in \text{Time}(M, [g])^{\kappa/2}$ . Set  $\kappa' := \kappa/2$ . Now the claim follows from lemma 7.8 in exactly the same fashion theorem 7.1 follows from lemma 7.2.  $\square$

*Proof of theorem 7.7.* Let  $\alpha \in (\mathfrak{T}^*)^\circ$  and  $\tau: \overline{M} \rightarrow \mathbb{R}$  be a calibration representing  $\alpha$ . By proposition 6.6 there exists  $\kappa = \kappa(\alpha) > 0$  such that  $v \in \text{Time}(M, [g])^\kappa$  for all  $v \in \mathfrak{V}(\tau)$ . Choose  $\varepsilon, \delta, \eta > 0$  and  $K' < \infty$  according to lemma 7.8. Assume that there exist  $v, w \in \mathfrak{V}(\tau)$  with

$$\text{dist}(\pi_{TM}(v), \pi_{TM}(w)) \leq \delta \text{ and } \text{dist}(v, w) \geq K' \text{dist}(\pi_{TM}(v), \pi_{TM}(w)).$$

Then lemma 7.8 implies that

$$d(\overline{\gamma}_v(-\varepsilon), \overline{\gamma}_w(\varepsilon)) + d(\overline{\gamma}_w(-\varepsilon), \overline{\gamma}_v(\varepsilon)) - L^g(\gamma_v|_{[-\varepsilon, \varepsilon]}) - L^g(\gamma_w|_{[-\varepsilon, \varepsilon]}) \geq \eta \text{dist}^2(v, w),$$

where  $\overline{\gamma}_v$  and  $\overline{\gamma}_w$  are lifts of  $\gamma_v$  resp.  $\gamma_w$  with  $\text{dist}(\overline{\gamma}_v(0), \overline{\gamma}_w(0)) = \text{dist}(\gamma_v(0), \gamma_w(0))$ . For  $\text{dist}(v, w) > 0$ , i.e.  $\gamma_v$  and  $\gamma_w$  do not coincide, this leads to a contradiction.

Since  $\gamma_v$  and  $\gamma_w$  are calibrated by  $\tau$  we have

$$\begin{aligned}\tau(\overline{\gamma}_v(\varepsilon)) - \tau(\overline{\gamma}_v(-\varepsilon)) &= \mathfrak{l}^*(\alpha)(d(\overline{\gamma}_v(-\varepsilon), \overline{\gamma}_v(\varepsilon))) = \mathfrak{l}^*(\alpha)L^g(\gamma_v|_{[-\varepsilon, \varepsilon]}), \\ \tau(\overline{\gamma}_w(\varepsilon)) - \tau(\overline{\gamma}_w(-\varepsilon)) &= \mathfrak{l}^*(\alpha)(d(\overline{\gamma}_w(-\varepsilon), \overline{\gamma}_w(\varepsilon))) = \mathfrak{l}^*(\alpha)L^g(\gamma_w|_{[-\varepsilon, \varepsilon]}), \\ \tau(\overline{\gamma}_w(\varepsilon)) - \tau(\overline{\gamma}_v(-\varepsilon)) &\geq \mathfrak{l}^*(\alpha)d(\overline{\gamma}_v(-\varepsilon), \overline{\gamma}_w(\varepsilon))\end{aligned}$$

and

$$\tau(\overline{\gamma}_v(\varepsilon)) - \tau(\overline{\gamma}_w(-\varepsilon)) \geq \mathfrak{l}^*(\alpha)d(\overline{\gamma}_w(-\varepsilon), \overline{\gamma}_v(\varepsilon)).$$

Then we get

$$\begin{aligned}0 &= [\tau(\overline{\gamma}_w(\varepsilon)) - \tau(\overline{\gamma}_v(-\varepsilon))] + [\tau(\overline{\gamma}_v(\varepsilon)) - \tau(\overline{\gamma}_w(-\varepsilon))] \\ &\quad - [\tau(\overline{\gamma}_v(\varepsilon)) - \tau(\overline{\gamma}_v(-\varepsilon))] - [\tau(\overline{\gamma}_w(\varepsilon)) - \tau(\overline{\gamma}_w(-\varepsilon))] \\ &\geq \mathfrak{l}^*(\alpha)d(\overline{\gamma}_v(-\varepsilon), \overline{\gamma}_w(\varepsilon)) + \mathfrak{l}^*(\alpha)d(\overline{\gamma}_w(-\varepsilon), \overline{\gamma}_v(\varepsilon)) \\ &\quad - \mathfrak{l}^*(\alpha)L^g(\gamma_v|_{[-\varepsilon, \varepsilon]}) - \mathfrak{l}^*(\alpha)L^g(\gamma_w|_{[-\varepsilon, \varepsilon]}) \\ &\geq \eta \mathfrak{l}^*(\alpha) \text{dist}^2(v, w) > 0.\end{aligned}$$

Note that for  $\text{dist}(v, w) = 0$  the claim is empty. This finishes the proof.  $\square$

For the proof of lemma 7.8 we will need the following theorem due to Weierstrass. For a discussion and proof in (the more general) time periodic case see [11]. Consider a Lagrange function  $\mathfrak{E}: TM \rightarrow \mathbb{R}$  with positive definite second fibre derivative and fibrewise superlinear growth. We say that a function  $\mathfrak{E}: TM \rightarrow \mathbb{R}$  has positive definite second fibre derivative if for any  $p \in M$  the restriction  $\mathfrak{E}|_{TM_p}$  has positive definite Hessian in any system of linear coordinates on  $TM_p$ . Further we say that  $\mathfrak{E}$  has fibrewise superlinear growth if

$$\frac{\mathfrak{E}(v)}{|v|} \rightarrow \infty \text{ as } |v| \rightarrow \infty, \text{ for all } v \in TM.$$

Define for an absolutely continuous curve  $\gamma: [a, b] \rightarrow M$  the *action*  $A^\mathfrak{E}$  of  $\gamma$  as  $A^\mathfrak{E}(\gamma) := \int_a^b \mathfrak{E}(\dot{\gamma}(t))dt$ .

**Theorem 7.9** (Weierstrass, [11]). *For any  $c > 0$ , there exist  $\varepsilon_0, C_0, C_1 > 0$ , such that if  $a < b \leq a + \varepsilon_0$  and  $\gamma: [a, b] \rightarrow M$  is a solution of the Euler-Lagrange equation satisfying  $|\dot{\gamma}(t)| \leq c$  for all  $t \in [a, b]$ , then*

$$A^\mathfrak{E}(\gamma_1) \geq A^\mathfrak{E}(\gamma) + F\left(\int_a^b \text{dist}(\dot{\gamma}(t), \dot{\gamma}_1(t))dt\right)$$

for any absolutely continuous curve  $\gamma_1: [a, b] \rightarrow M$  such that  $\gamma_1(a) = \gamma(a)$  and  $\gamma_1(b) = \gamma(b)$ . Here,

$$F(s) = \min\{C_0 s, C_1 s^2\}.$$

Moreover, still assuming  $b - a \leq \varepsilon_0$ , we have that for any  $x_a, x_b \in M$  such that  $\text{dist}(x_a, x_b) \leq c(b - a)/2$ , there exists a solution  $\gamma$  of the Euler-Lagrange equation satisfying  $\gamma(a) = x_a, \gamma(b) = x_b$ , and  $|\dot{\gamma}(t)| \leq c$ , for all  $t \in [a, b]$ .

**Lemma 7.10** ([11]). *If  $c > 0$ , then there exist  $\varepsilon, \delta, \eta, K' > 0$  such that if  $x_{1,2}: [-\varepsilon, \varepsilon] \rightarrow M$  are solutions of the Euler-Lagrange equation of  $\mathfrak{E}$  with*

$$|\dot{x}_i(0)| \leq c, \text{dist}(x_1(0), x_2(0)) \leq \delta \text{ and } \text{dist}(\dot{x}_1(0), \dot{x}_2(0)) \geq K' \text{dist}(x_1(0), x_2(0)),$$

then there exist  $C^1$ -curves  $y_1, y_2: [-\varepsilon, \varepsilon] \rightarrow M$  such that  $y_1(-\varepsilon) = x_1(-\varepsilon), y_1(\varepsilon) = x_2(\varepsilon), y_2(-\varepsilon) = x_2(-\varepsilon), y_2(\varepsilon) = x_1(\varepsilon)$  and

$$A^\mathfrak{E}(x_1) + A^\mathfrak{E}(x_2) - A^\mathfrak{E}(y_1) - A^\mathfrak{E}(y_2) \geq \eta \text{dist}^2(\dot{x}_1(0), \dot{x}_2(0)).$$



*Proof of Lemma 7.8.* The idea is to transform the problem to fit the situation of lemma 7.10. Choose for every  $\bar{\varepsilon} < \frac{\text{inj}(M,g)}{3}$  a real number  $\bar{\delta} = \bar{\delta}(\bar{\varepsilon}) \in (0, \bar{\varepsilon})$  such that  $B_{\bar{\delta}}(\chi(0)) \subseteq I_U^+(\chi(-\bar{\varepsilon}))$  for all future pointing pregeodesics  $\chi: \mathbb{R} \rightarrow M$  with  $\chi'(0) \in \text{Time}(M, [g])^\kappa$ , where  $U$  is any convex normal neighborhood of  $B_{2\bar{\varepsilon}}(\chi(0))$ . Next choose  $\bar{\kappa} \in (0, \kappa)$  for the pair  $(\bar{\varepsilon}, \bar{\delta})$  such that for any pair of future pointing pregeodesics  $\chi_1, \chi_2: \mathbb{R} \rightarrow M$  with  $\text{dist}(\chi_1(0), \chi_2(0)) \leq \bar{\delta}$  and  $\chi_1'(0), \chi_2'(0) \in \text{Time}(M, [g])^\kappa$  the unique pregeodesic  $\psi: [-\varepsilon', \varepsilon'] \rightarrow M$  with  $\psi(-\varepsilon') = \chi_1(-\bar{\varepsilon})$  and  $\psi(\varepsilon') = \chi_2(\bar{\varepsilon})$  satisfies  $\psi'(t) \in \text{Time}(M, [g])^{\bar{\kappa}}$  for all  $t \in [-\varepsilon', \varepsilon']$ .

Set

$$\mathfrak{E}': \text{Time}(M, [g])^{\bar{\kappa}/2} \cap T^{1,R}M \rightarrow \mathbb{R}, v \mapsto -\sqrt{|g(v, v)|}$$

$\mathfrak{E}'$  is a convex function w.r.t. to the induced Riemannian metric on  $\text{Time}(M, [g]) \cap T^{1,R}M$  and has positive definite second fibre derivative everywhere. Choose a convex extension  $\mathfrak{E}: TM \rightarrow \mathbb{R}$  of  $\mathfrak{E}'$  such that the second fibre derivative is positive definite,  $\mathfrak{E}$  has superlinear growth and

$$(12) \quad -\sqrt{|g(v, v)|} \leq \mathfrak{E}(v)$$

for all future pointing  $v \in TM$ . For an absolutely continuous curve  $\gamma: [a, b] \rightarrow M$  set  $A^{\mathfrak{E}}(\gamma) := \int_a^b \mathfrak{E}(\dot{\gamma}(t)) dt$ . Note that under these conditions there exists  $\varepsilon_1 = \varepsilon_1(g, g_R, \mathfrak{E}) > 0$  such that every pregeodesic  $x: [-\varepsilon_1, \varepsilon_1] \rightarrow M$  with  $x'(t) \in \text{Time}(M, [g])^{\bar{\kappa}}$  for all  $t \in [-\varepsilon_1, \varepsilon_1]$  is a minimizers of  $\mathfrak{E}$ .

More precisely, choose  $\varepsilon_0 > 0$  for  $c = 1$  according to theorem 7.9 and consider a pregeodesic  $x: [a, b] \rightarrow M$  with  $x'(t) \in \text{Time}(M, [g])^{\bar{\kappa}}$  for all  $t \in [a, b]$  and  $b - a \leq \varepsilon_0$ . Since  $x$  is parameterized w.r.t.  $g_R$ -arclength we have  $\text{dist}(x(a), x(b)) \leq \varepsilon_0$ . By theorem 7.9 there exists a solution  $y: [a, b] \rightarrow M$  of the Euler-Lagrange equation of  $\mathfrak{E}$  with  $y(a) = x(a)$ ,  $y(b) = x(b)$  and  $|\dot{y}(t)| \leq 1$  for all  $t \in [a, b]$ . This solution is a minimizer according to theorem 7.9. Using the Taylor expansion of  $x$  and  $y$  in a system of local coordinates and noting that  $x$  as well as  $y$  satisfy an ordinary differential equation of second order with locally bounded coefficients, we see that

$$\text{dist}(x'(a), \dot{y}(a)) \leq C(b - a)$$

for some  $C < \infty$  depending only on  $g, g_R$  and  $\mathfrak{E}$ . For  $b - a \leq \frac{\bar{\kappa}}{2C}$  we have  $\dot{y}(a) \in \text{Time}(M, [g])^{\bar{\kappa}/2}$ , since we assumed  $x'(a) \in \text{Time}(M, [g])^{\bar{\kappa}}$ . With the continuity of the Euler-Lagrange flow of  $\mathfrak{E}$  we obtain that  $y$  is future pointing for sufficiently small  $b - a \leq \min\{\varepsilon_0, \frac{\bar{\kappa}}{2C}\}$ . Since  $x$  locally maximizes  $g$ -arclength we have

$$A^{\mathfrak{E}}(y) \geq -L^g(y) \geq -L^g(x) = A^{\mathfrak{E}}(x),$$

by (12), and the pregeodesic  $x$  is identical with the minimizer  $y$  according to theorem 7.9.

According to lemma 7.10, there exist  $\varepsilon, \delta, \eta > 0$  and  $K' < \infty$  such that if  $\text{dist}(x_1(0), x_2(0)) \leq \delta$  and  $\text{dist}(x_1'(0), x_2'(0)) \geq K' \text{dist}(x_1(0), x_2(0))$  we have

$$A^{\mathfrak{E}}(x_1) + A^{\mathfrak{E}}(x_2) - A^{\mathfrak{E}}(y_1) - A^{\mathfrak{E}}(y_2) \geq \eta \text{dist}(\dot{x}_1(0), \dot{x}_2(0))^2,$$

for the  $\mathfrak{E}$ -minimizer  $y_1, y_2: [-\varepsilon, \varepsilon] \rightarrow M$  with  $y_1(-\varepsilon) = x_1(-\varepsilon)$ ,  $y_1(\varepsilon) = x_2(\varepsilon)$ ,  $y_2(-\varepsilon) = x_2(-\varepsilon)$  and  $y_2(\varepsilon) = x_1(\varepsilon)$ .

It remains to show that the curves  $y_1, y_2$  are future pointing for  $\varepsilon, \delta > 0$  sufficiently small. W.l.o.g. we can assume that  $\varepsilon \leq \bar{\varepsilon}$  and  $\delta \leq \bar{\delta}$ . Choose a convex normal neighborhood  $U$  of  $x_1(0)$  with  $B_{2\varepsilon+\delta}(x_1(0)) \subseteq U$ . Then we have  $x_1, x_2 \subseteq U$ . For the unique pregeodesics  $\psi_{1,2}: [-\varepsilon'_{1,2}, \varepsilon'_{1,2}] \rightarrow U$  such that  $\psi_1(-\varepsilon'_1) = x_1(-\varepsilon)$ ,  $\psi_1(\varepsilon'_1) = x_2(\varepsilon)$ ,  $\psi_2(-\varepsilon'_2) = x_2(-\varepsilon)$  and  $\psi_2(\varepsilon'_2) = x_1(\varepsilon)$  we have  $\psi'_i(t) \in \text{Time}(M, [g])^{\bar{\kappa}}$  for all  $|t| \leq \varepsilon'_i$  by our assumption on  $(\bar{\varepsilon}, \bar{\delta})$ . We have seen above that the minimizer  $y_i: [-\varepsilon'_i, \varepsilon'_i] \rightarrow M$  with  $y_i(\pm\varepsilon'_i) = \psi_i(\pm\varepsilon'_i)$  is identical to  $\psi_i$  for  $\varepsilon'_i$  sufficiently small. Since we know that  $\varepsilon'_i \leq C_{g,g_R} \varepsilon$  (Corollary 2.12), the bound on  $\varepsilon'_i$  depends only on

$\kappa$ ,  $g$  and  $g_R$ . Using (12) we have  $A^{\mathfrak{E}}(y_i) \geq -L^g(y_i)$ . Since  $A^{\mathfrak{E}}(x_i) = -L^g(x_i)$  the lemma follows immediately.  $\square$

## 8. THE HEDLUND EXAMPLES

In [10] Hedlund gave an example of a Riemannian 3-torus to show that his results on closed geodesics in Riemannian 2-tori do not generalize to higher dimensions. Bangert then employed the idea in [2] to construct a class of Riemannian metrics on 3-tori (called Hedlund examples) to show the optimality of his results. We aim for the same goal with our construction.

Consider  $\mathbb{R}^3$  together with the standard basis  $\{e_1, e_2, e_3\}$  and a system of straight lines and neighborhoods as in [2]. More precisely, let  $l_1 := \mathbb{R} \times \{0\} \times \{0\}$ ,  $l_2 := \{0\} \times \mathbb{R} \times \{\frac{1}{2}\}$  and  $l_3 := \{\frac{1}{2}\} \times \{\frac{1}{2}\} \times \mathbb{R}$  (not to be mistaken for the stable time separation  $l$ ). Set  $L_1 := l_1 + \mathbb{Z}^3$ ,  $L_2 := l_2 + \mathbb{Z}^3$ ,  $L_3 := l_3 + \mathbb{Z}^3$  and  $L := L_1 \cup L_2 \cup L_3$ . Denote the coordinate functions relative to  $\{e_1, e_2, e_3\}$  by  $x^1$ ,  $x^2$  and  $x^3$ . Further choose the canonical flat metric  $g_R := \sum dx_i^2$  as the Riemannian background metric on  $\mathbb{R}^3$ . Next let  $\{v_1 := \frac{1}{\sqrt{3}}(1, 1, 1), v_2, v_3\}$  be a orthonormal basis of  $\mathbb{R}^3$  with respect to the standard Euclidian scalar product and let  $\{v_1^*, v_2^*, v_3^*\}$  be the dual basis. Define for  $\lambda_i > 0$  ( $i \in \{1, 2, 3\}$ ) with  $\sum \lambda_i = 1$  and  $\varepsilon \in (0, 10^{-2})$  the Lorentzian metrics

$$\begin{aligned} g_\varepsilon &:= -\frac{\varepsilon^2}{4} v_1^* \otimes v_1^* + v_2^* \otimes v_2^* + v_3^* \otimes v_3^*, \\ g_1 &:= (\lambda_1)^2 \left( -(dx^1)^2 + \frac{1}{3}(dx^2)^2 + \frac{1}{3}(dx^3)^2 \right), \\ g_2 &:= (\lambda_2)^2 \left( \frac{1}{3}(dx^1)^2 - (dx^2)^2 + \frac{1}{3}(dx^3)^2 \right) \\ \text{and} \\ g_3 &:= (\lambda_3)^2 \left( \frac{1}{3}(dx^1)^2 + \frac{1}{3}(dx^2)^2 - (dx^3)^2 \right). \end{aligned}$$

Consider a  $\mathbb{Z}^3$ -invariant Lorentzian metric  $\bar{g}$  on  $\mathbb{R}^3$  such that the following three conditions are satisfied:

- (i)  $\bar{g}_p \geq g_\varepsilon$  for all  $p \in \mathbb{R}^3$ .
- (ii)  $g_{2\varepsilon} \geq \bar{g}_p$  for  $p \in \mathbb{R}^3 \setminus B_\varepsilon(L)$ .
- (iii) For  $p \in B_\varepsilon(L_i)$  we have  $g_i \geq \bar{g}_p$  with equality exactly on  $L_i$ .

$\bar{g}$  naturally induces a Lorentzian metric  $g$  on  $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ . By condition (i),  $v_1$  is everywhere timelike and thus can be used to time orient on  $(T^3, g)$ .  $(T^3, g)$  is vicious by (i). Further  $(\mathbb{R}^3, \bar{g})$  is globally hyperbolic since  $v_1^*$  is a smooth uniform temporal function on  $(\mathbb{R}^3, \bar{g})$ , i.e.  $\nabla^{\bar{g}} v_1^*$  is a smooth vector field with  $|\nabla^{\bar{g}} v_1^*|$  uniformly bounded away from 0 and  $\infty$ .

The conditions (i)-(iii) have the following immediate consequences:

- (1) The straight lines in  $L_i$  are  $\bar{g}$ -future pointing timelike maximal geodesics. The  $\bar{g}$ -length of a segment on such a line is exactly  $\alpha_i x^i$ .
- (2) For two neighboring lines  $l_i, l_j$  in  $L$ , i.e.  $\text{dist}(l_i, l_j) = 1/2$ , the Riemannian length of any causal curve connecting  $\partial B_\varepsilon(l_i)$  with  $\partial B_\varepsilon(l_j)$  is bounded from above by  $\frac{1}{2} - 2\varepsilon$ .

For the second observation first note that any causal curves in  $(\mathbb{R}^3, \bar{g})$  contained in the complement of  $B_\varepsilon(L)$  and connecting two points  $p$  and  $q$ , must be contained in the  $\varepsilon|q - p|$ -neighborhood of the straight line segment between  $p$  and  $q$ . Second, the distance of  $q - p/|q - p|$  from  $\frac{1}{\sqrt{3}}(1, 1, 1)$  is bounded by  $2\varepsilon$ . Now for two given lines  $l_i$  and  $l_j$  in  $L$  with  $\text{dist}(l_i, l_j) \leq 1/2$ , there exists exactly one line segment with

direction  $(1, 1, 1)$  and endpoints in  $l_i \cup l_j$ . Now by the previous observations, any causal curve with endpoints in  $B_\varepsilon(l_i) \cup B_\varepsilon(l_j)$  is contained in the  $2\varepsilon$ -neighborhood of this line segment.

The Riemannian length of future pointing curves can be estimated in the sense of corollary 2.12.

**Fact 8.1.** *Let  $p, q \in \mathbb{R}^3$  and  $\gamma: I \rightarrow \mathbb{R}^3$  a future pointing curve between  $p$  and  $q$ . Then*

$$L^{g_R}(\gamma) \leq 2 \left( \sum (q - p)^i + 4\varepsilon \right).$$

*Proof.* Assume  $\gamma$  to be parameterized by  $g_R$ -arclength. Set  $A' := \gamma^{-1}(\mathbb{R}^3 \setminus B_\varepsilon(L))$  and  $A'_i := \gamma^{-1}(B_\varepsilon(L_i))$ .

By (ii) and (iii) we know that  $|w|^2 \leq 4(w^i)^2$  for every causal  $w \in T\mathbb{R}_z^3$  with  $z \in \mathbb{R}^3 \setminus B_\varepsilon(L_j \cup L_k)$  and  $\{i, j, k\} = \{1, 2, 3\}$ . Then we have  $L^{g_R}(\gamma|_{A' \cup A'_i}) \leq \max\{2(q^i - p^i), 0\}$ .

If  $(q - p)^i \leq 0$ , there exists a line  $l \in L \setminus L_i$  such that  $\gamma(I) \subseteq B_{6\varepsilon}(l)$ . Note that for every  $z \in \mathbb{R}^3 \setminus B_\varepsilon(L)$ , every future pointing vector  $w \in T\mathbb{R}_z^3$  and every  $i \in \{1, 2, 3\}$  we have  $w^i \geq (\frac{1}{\sqrt{3}} - \varepsilon)|w|$ . But then  $x^i(z) \geq x^i(l) + \varepsilon$  for any point  $z \in \partial B_{6\varepsilon}(l) \cap J^+(B_\varepsilon(l))$ . By condition (ii),  $x^i$  cannot decrease along  $\gamma$  near  $t$  if  $\gamma(t) \notin B_\varepsilon(l)$ . Therefore  $(q - p)^i > -2\varepsilon$  and consequently  $L^{g_R}(\gamma) \leq 2(q - p)^j$  for  $l \subseteq L_j$ . In general we obtain

$$L^{g_R}(\gamma) \leq 2 \left( \sum (q - p)^i + 4\varepsilon \right).$$

□

**Proposition 8.2.** *For  $(T^3, g)$  as above we have  $\mathfrak{T} = \text{pos}\{e_1, e_2, e_3\}$ .*

*Proof.* Use the fact, mentioned already in the proof above, that  $p + h \in J^+(p)$  implies  $h^i \geq -2\varepsilon$  for  $i \in \{1, 2, 3\}$ . Therefore we have  $\mathfrak{T} \subseteq \text{pos}\{e_1, e_2, e_3\}$ . The other inclusion follows from the fact that the curves  $p + te_i$  are future pointing timelike if  $p \in L_i$ . □

The next step is the construction of the so-called *standard paths*. Standard paths were introduced in [2] in the construction of the Hedlund examples. The standard-paths in the present work are almost identical to those in [2]. The main difference is that we have to take care that we construct future pointing curves.

Let  $p, p + h \in L$  with  $h^1, h^2, h^3 \geq 1/2$ ,  $p \in l_i \subseteq L_i$ ,  $p + h \in l_j \subseteq L_j$  and  $j \neq i$ . Then the standard-path from  $p$  to  $p + h$  is defined as follows:

First assume that  $h^k \geq 1$  for  $k \neq i, j$ . Define  $l_k \subseteq L_k$ ,  $k \neq i, j$ , to be the unique line with  $x^j(l_i) < x^j(l_k)$ ,  $x^i(l_k) < x^i(l_j)$  and  $\text{dist}(l_i, l_k) = \text{dist}(l_k, l_j) = 1/2$ . The conditions imposed on  $l_i, l_j$  and  $l_k$  imply that the points  $p_i \in l_i$  and  $p_k \in l_k$  with  $x^i(p_i) = x^i(l_k) - 1/2$  and  $x^j(p_k) = x^j(l_j) - 1/2$  are uniquely determined. Further we have  $p_i + \frac{2}{\sqrt{3}}v_1 \in l_k$  and  $p_k + \frac{2}{\sqrt{3}}v_1 \in l_j$ .

Now a standard path from  $p$  to  $p + h$  consists of following  $l_i$  from  $p$  until  $p_i$ , changing to  $l_k$ , by following the straight line segment with direction  $v_1$  to  $l_k$ , then following  $l_k$  until  $p_k$ , changing to  $l_j$  via the line segment with direction  $v_1$  and finally following  $l_j$  until  $p + h$ .

For  $h^k = 1/2$  follow  $l_i$  until  $p_i$  with  $x^i(p_i) = x^i(l_j) - 1/2$ , then change to  $l_j$  and follow  $l_j$  until  $p + h$ .

This especially implies  $q \in J^+(p)$  for all  $p, q \in L$  with  $(q - p)^i \geq 1/2$  for  $i \in \{1, 2, 3\}$ .

To illustrate proposition 2.13, we note the following proposition.

**Proposition 8.3.** *We have  $q \in J^+(p)$  for all  $p, q \in \mathbb{R}^3$  with  $(q - p)^i \geq \frac{1}{\varepsilon} + \frac{3}{2}$  for all  $i \in \{1, 2, 3\}$ .*

*Proof.* By condition (i) for any pair of points  $p, q \in \mathbb{R}^3$  there exist straight lines  $l \subseteq L_j$  intersecting  $B_{\frac{1}{2\varepsilon}}(p) \cap J^+(p)$  and  $l' \subseteq L_k$  intersecting  $B_{\frac{1}{2\varepsilon}}(q) \cap J^-(q)$  with  $j \neq k$ . Points  $p' \in l$  and  $q' \in l'$  are connectable via standard paths if  $(q' - p')^i \geq 1/2$  for  $i \in \{1, 2, 3\}$ .  $\square$

**Proposition 8.4.** *The stable time separation of  $(T^3, g)$  is given by*

$$\mathfrak{l}(h) = \sum \lambda_i h^i$$

for  $h \in \text{pos}\{e_1, e_2, e_3\}$ .

In order to give a proof we have to make some technical statements. Following [2], a future pointing curve  $\gamma: I \rightarrow \mathbb{R}^3$  is said to change tubes  $n$  times if there exist parameter values  $t_0 < t_1 < \dots < t_n \in I$  such that  $\gamma(t_{i-1})$  and  $\gamma(t_i)$  lie in different components (i.e. tubes) of  $B_\varepsilon(L)$ .

Denote the endpoints of  $\gamma$  with  $p$  and  $p + h$ . For  $i \in \{1, 2, 3\}$  consider the closed set  $\gamma^{-1}(\overline{B_\varepsilon(L_i)})$ . Denote by  $B_{i,k}$  the connected component of the complement of  $\gamma^{-1}(\overline{B_\varepsilon(L_i)})$  in  $I$  whose boundary points belong to the same  $\gamma^{-1}(B_\varepsilon(l_i + k))$  for some  $k \in \mathbb{Z}^3$ . Define

$$A_i := \gamma^{-1}(\overline{B_\varepsilon(L_i)}) \cup (\cup_{k \in \mathbb{Z}^3} B_{i,k}).$$

Now the connected components of the set  $A := I \setminus (A_1 \cup A_2 \cup A_3)$  correspond exactly to those arcs of  $\gamma$  on which  $\gamma$  either changes tubes or the initial and final arcs of  $\gamma$  outside the tubes.

**Lemma 8.5.** *Let  $p, q \in \mathbb{R}^3$  and  $\gamma: I \rightarrow \mathbb{R}^3$  be a future pointing curve connecting  $p$  with  $q$ . Set  $A$  as before. Then we have*

$$\sum \lambda_i \int_A \dot{\gamma}^i \leq (1 - 8\varepsilon) \left( \sum \lambda_i (q - p)^i - L^g(\gamma) + 4\varepsilon \right).$$

*Proof.* First observe that for  $i \in \{1, 2, 3\}$ ,  $p \in B_\varepsilon(L_i)$  and  $v \in T\mathbb{R}_p^3$  a future pointing vector, we have  $\sqrt{|\overline{g}_p(v, v)|} \leq \lambda_i v^i$ . Next, if  $v \in T\mathbb{R}^3$  is future pointing for  $g_\varepsilon$  we have  $\sqrt{|g_\varepsilon(v, v)|} \leq \varepsilon \sum \lambda_i v^i$  (Note that  $\sqrt{|g_\varepsilon(v, v)|} \leq \frac{\varepsilon}{2}|v|$ ,  $|\frac{v^i}{|v|} - \frac{1}{\sqrt{3}}| \leq \frac{\varepsilon}{2}$  and  $\sum \lambda_i = 1$ ).

Let  $i \in \{1, 2, 3\}$ . For each connected component  $C(A_j)$  of  $A_j$  both endpoints are endpoints of connected components of  $A$  or contain at least one endpoint of  $I$ . For  $i \neq j$  and an adjacent component  $C(A)$  of  $A$  we have

$$\int_{C(A_j)} \dot{\gamma}^i \geq \frac{-2\varepsilon}{\frac{1}{2} - 2\varepsilon} \int_{C(A)} \dot{\gamma}^i \text{ or } \int_{C(A_j)} \dot{\gamma}^i \geq -2\varepsilon.$$

Since  $C(A)$  can be adjacent to two different components of  $A_1 \cup A_2 \cup A_3$  we conclude

$$\int_{A_j \cup A_k} \dot{\gamma}^i \geq -\frac{8\varepsilon}{1 - 4\varepsilon} \int_A \dot{\gamma}^i - 4\varepsilon$$

for  $\{i, j, k\} = \{1, 2, 3\}$ . Now we estimate

$$\begin{aligned} L^g(\gamma) &= L^g(\gamma|_A) + \sum L^g(\gamma|_{A_i}) \leq \varepsilon \sum \lambda_i \int_A \dot{\gamma}^i + \sum \lambda_i \int_{A_i} \dot{\gamma}^i \\ &\leq \sum \lambda_i (q - p)^i - \left(1 - \frac{8\varepsilon}{1 - 4\varepsilon} - \varepsilon\right) \sum \lambda_i \int_A \dot{\gamma}^i + 4\varepsilon \\ &\leq \sum \lambda_i (q - p)^i - \frac{1}{1 - 8\varepsilon} \sum \lambda_i \int_A \dot{\gamma}^i + 4\varepsilon. \end{aligned}$$

$\square$

To complete the proof of proposition 8.4, we use the standard-paths and proposition 8.3 to estimate the time separation  $d(p, q)$  between any two  $p, q \in \mathbb{R}^3$  with  $(q - p)^i \geq \frac{1}{\varepsilon} + \frac{3}{2}$ . We have

$$(13) \quad d(p, q) \geq \sum \lambda_i (q - p)^i - \left( \frac{1}{\varepsilon} + \frac{3}{2} \right) \sum \lambda_i.$$

If  $q - p \in \partial \mathfrak{T}$ , say  $(q - p)^k = 0$ , choose points  $p', q' \in L$  with  $(q')^k = (p')^k + \frac{1}{2}$  and  $\text{dist}(q - p, q' - p') \leq \sqrt{2}$ . A standard-path connecting  $p'$  and  $q'$  yields the result. Notice the trivial estimate  $d(p, q) \leq \sum \lambda_i (q - p)^i$  for  $q - p \in \text{pos}\{e_1, e_2, e_3\}$ . This completes the proof of proposition 8.4.

### 8.1. Timelike Maximizers.

**Proposition 8.6.** *A maximal future pointing geodesic segment  $\gamma: [a, b] \rightarrow \mathbb{R}^3$  with endpoints  $p \in L_i$  and  $q \in L_j$  ( $i \neq j$ ) lies at a Riemannian distance of at most  $4\varepsilon$  from the standard-path connecting  $p$  and  $q$ .*

*Proof.* We have

$$d(p, q) \geq \sum \lambda_i (q - p)^i - \sum \lambda_i,$$

if  $x^k(l_i) > x^k(l_j) + 1$  and  $\{i, j, k\} = \{1, 2, 3\}$ . Analogously we obtain

$$d(p, q) \geq \sum \lambda_i (q - p)^i - \frac{1}{2} \sum \lambda_i,$$

if  $x^k(l_i) = x^k(l_j) + 1/2$  ( $\{i, j, k\} = \{1, 2, 3\}$ ).

Recall the definition of  $A$  from above. Let  $\sharp A$  be the number of connected components of  $A$ . Then we have  $\int_A \dot{\gamma}^i \geq \sharp A (\frac{1}{2} - 2\varepsilon)$  for all  $i \in \{1, 2, 3\}$ . Consequently maximizers can change tubes only twice in the first case and once in the second case. The proposition follows from the observation that for  $l \subseteq L_i$ ,  $l' \subseteq L_j$  with  $x^k(l) \leq x^k(l')$  ( $\{i, j, k\} = \{1, 2, 3\}$ ) and  $\text{dist}(l, l') = 1/2$ , the intersection  $J^+(l) \cap J^-(l')$  is contained in  $B_{4\varepsilon}(x + \text{span}\{v_1\})$ , where  $x \in \mathbb{R}^3$  is the unique point in  $l$  with  $x^i(x) = x^i(l') - 1/2$ .  $\square$

**Proposition 8.7.** *A maximal geodesic segment  $\gamma: [a, b] \rightarrow \mathbb{R}^3$  can change tubes at most six times.*

*Proof.* It suffices to consider future pointing geodesics. Set  $a' := \inf \gamma^{-1}(B_\varepsilon(L))$  and  $b' := \sup \gamma^{-1}(B_\varepsilon(L))$ . Choose  $l, l' \subseteq L$  with  $\gamma(a') \in B_\varepsilon(l)$  and  $\gamma(b') \in B_\varepsilon(l')$ . Then the intersections  $J^+(\gamma(a')) \cap (l + (1, 1, 1))$  and  $J^-(\gamma(b')) \cap (l - (1, 1, 1))$  are nonempty. Note that we can choose points in  $p \in J^+(\gamma(a')) \cap (l + (1, 1, 1))$  and  $q \in J^-(\gamma(b')) \cap (l - (1, 1, 1))$  with  $\text{dist}(\gamma(a'), p)$  resp.  $\text{dist}(\gamma(b'), q) \leq \sqrt{3} + 2\varepsilon$ . We obtain

$$\begin{aligned} d(\gamma(a'), \gamma(b')) &\geq d(p, q) \geq \sum \lambda_i (q - p)^i - \sum \lambda_i \\ &\geq \sum \lambda_i (\gamma(b') - \gamma(a'))^i - 3 \sum \lambda_i. \end{aligned}$$

With lemma 8.5 we conclude

$$\left( \frac{1}{2} - 2\varepsilon \right) \sharp A \leq (1 - 8\varepsilon) \left( 3 \sum \lambda_i + 4\varepsilon \right).$$

$\square$

**Corollary 8.8.** *Every maximal geodesic has asymptotic distance in each of its senses to one of the lines in  $L$  of at most  $\varepsilon$ .*

*Proof.* Like before we can assume all curves to be future pointing. Let  $\gamma: [a, b] \rightarrow (\mathbb{R}^3, \overline{g})$  be maximal. If  $\gamma$  never intersects  $B_\varepsilon(L)$ , we have  $L^g(\gamma) \leq \varepsilon \sum \lambda_i(\gamma^i(b) - \gamma^i(a))$ . From (13) we know that  $L^g(\gamma) \geq \sum \lambda_i((\gamma^i(b) - \gamma^i(a)) - (\frac{1}{\varepsilon} + \frac{3}{2}))$ . Consequently

$$0 \leq \sum \lambda_i \left( (\varepsilon - 1)(\gamma(b) - \gamma(a))^i + \left( \frac{1}{\varepsilon} + \frac{3}{2} \right) \right).$$

If  $\text{dist}(\gamma(a), \gamma(b)) \geq \frac{1}{\varepsilon} \geq \sqrt{3} \frac{1+3\varepsilon}{2\varepsilon(1-\varepsilon)(1-\sqrt{3}\varepsilon)}$  and  $\gamma$  does not intersect  $B_\varepsilon(L)$ , then  $\gamma$  cannot be maximal.  $\square$

**Proposition 8.9.** *For each pair of future pointing lines  $l \subseteq L_i$ ,  $l' \subseteq L_j$  ( $i \neq j$ ) with  $x^k(l') \geq x^k(l)$ ,  $\{i, j, k\} = \{1, 2, 3\}$ , there exists a maximal geodesic  $\gamma$  which is asymptotic to  $l'$  for  $t \rightarrow \infty$  and asymptotic to  $l$  for  $t \rightarrow -\infty$ .*

**Remark 8.10.** *For  $i \neq j$  and  $l \subseteq L_i$ ,  $l' \subseteq L_j$  either  $l' \cap J^+(l) \neq \emptyset$  or  $l \cap J^+(l') \neq \emptyset$ , depending on whether  $x^k(l') \geq x^k(l)$  or  $x^k(l) \geq x^k(l')$  for  $\{i, j, k\} = \{1, 2, 3\}$ . For  $l, l' \in L_i$  we have  $l' \in J^+(l)$  iff  $x^j(l') > x^j(l)$  and  $x^k(l') > x^k(l)$  for  $\{i, j, k\} = \{1, 2, 3\}$ .*

**Lemma 8.11.** *There exists  $\varepsilon' \in (0, \varepsilon]$  such that for all  $\delta \in (0, \varepsilon')$  there exists  $r(\delta) < \infty$  such that every future pointing maximal pregeodesic  $\gamma: [a, b] \rightarrow \mathbb{R}^3$  with  $|\dot{\gamma}| \equiv 1$  and endpoints in a tube  $B_{\varepsilon'}(l)$  for some  $l \subseteq L$  satisfies  $\gamma(s) \in B_\delta(l)$  for  $s \in [a + r(\delta), b - r(\delta)]$ .*

*Proof.* Let  $l \subseteq L_j$ . Choose  $\varepsilon' \in (0, \varepsilon]$  such that

$$B_{\varepsilon'}(l) \subseteq \{p \in B_\varepsilon(l) \mid g_p \geq \frac{\lambda_j^2}{3}(-(dx^j)^2 + (dx^i)^2 + (dx^k)^2)\}.$$

Denote for  $p \in B_{\varepsilon'}(l)$  by  $p' \in l$  the Euclidian orthogonal projection of  $p$  onto  $l$ . Then the curve  $t \in [0, 1] \mapsto p + t(|p - p'|e_j + (p' - p))$  is future pointing. Consequently for all  $\delta \in (0, \varepsilon']$  and all  $p, q \in B_\delta(l)$  we have

$$(14) \quad d(p, q) \geq \lambda_j(q - p)^j - 2\lambda_j\delta.$$

Set  $A_\delta := \gamma^{-1}(B_\delta(l))$  and choose  $\eta(\delta) \in (0, \lambda_j/2)$  such that

$$\sqrt{|\overline{g}_p(v, v)|} \leq (\lambda_j - \eta(\delta))v^j$$

for any  $p \in B_\varepsilon(l) \setminus B_\delta(l)$  and any future pointing vector  $v \in T\mathbb{R}_p^3$  (recall condition (iii)). Note that a future pointing curve with endpoints in  $B_\varepsilon(l)$  cannot intersect a different  $B_\varepsilon(l')$ . We have

$$L^g(\gamma) \leq (\lambda_j - \eta(\delta)) \int_{A_\delta^c} \dot{\gamma}^j + \lambda_j \int_{A_\delta} \dot{\gamma}^j = \lambda_j(\gamma(b) - \gamma(a))^j - \eta(\delta) \int_{A_\delta} \dot{\gamma}^j.$$

On the other hand the maximality of  $\gamma$  implies  $L^g(\gamma) \geq \lambda_j(\gamma(b) - \gamma(a))^j - 2\lambda_j\varepsilon'$  and thus we obtain

$$\int_{A_\delta^c} \dot{\gamma}^j \leq \frac{2\lambda_j\varepsilon'}{\eta(\delta)}.$$

Set  $\delta' := \eta(\frac{\delta}{2})\frac{\delta}{2}$  and  $r(\delta) := \frac{4\varepsilon'\lambda_j}{\eta(\delta')}$ . Note that for all  $p \in (B_\varepsilon(L_i) \cup B_\varepsilon(L_k))^c$  ( $\{i, j, k\} = \{1, 2, 3\}$ ) and all future pointing vectors  $v \in T\mathbb{R}_p^3$  we have  $|v|^2 \leq \frac{4}{3}(v^j)^2$ . Then by the previous argument there exist  $s^- \in [a, a + r(\delta)]$  and  $s^+ \in [b - r(\delta), b]$  with  $\gamma(s^\pm) \in B_{\delta'}(l)$ . To complete the proof assume there exists  $s \in [a + r(\delta), b -$

$r(\delta)]$  with  $\gamma(s) \in B_\delta(l)^c$ . With (14) we know that  $d(\gamma(s^-), \gamma(s^+)) \geq \lambda_j(\gamma^j(s^+) - \gamma^j(s^-)) - 2\lambda_j\delta'$ . Consequently

$$\begin{aligned} L^g(\gamma|_{[s^-, s^+]}) &\leq \left( \lambda_j - \eta \left( \frac{\delta}{2} \right) \right) \int_{A_{\delta/2}^c \cap [s^-, s^+]} \dot{\gamma}^j + \lambda_j \int_{A_{\delta/2} \cap [s^-, s^+]} \dot{\gamma}^j \\ &= \lambda_j(\gamma^j(s^+) - \gamma^j(s^-)) - \eta \left( \frac{\delta}{2} \right) \int_{[s^-, s^+] \setminus A_{\delta/2}} \dot{\gamma}^j. \end{aligned}$$

Since  $\int_{[s^-, s^+] \setminus A_{\delta/2}} \dot{\gamma}^j \geq \delta$  we have  $\delta' > \eta \left( \frac{\delta}{2} \right) \frac{\delta}{2}$ . This contradicts the choice of  $\delta'$ .  $\square$

*Proof of proposition 8.9.* Let  $x \in l$  with  $x^i(x) = x^i(l')$  and  $x' \in l'$  with  $x^j(x') = x^j(l)$ . The assumption  $x^k(l') \geq x^k(l)$  implies that the standard-path from  $x - ne_i$  to  $x' + ne_j$  is defined for all  $n \in \mathbb{N}$  (compare previous remark).

With proposition 8.6 we know that a maximal geodesic  $\gamma_n$  from  $x - ne_i$  to  $x' + ne_j$  stays within a distance of  $4\varepsilon$  from the standard-path between  $x - ne_i$  and  $x' + ne_j$ . Recall that we can estimate the length of the standard-path, and therefore the time separation of  $x - ne_i$  and  $x' + ne_j$ , by

$$L^g(\gamma_n) \geq \sum_{\tau=1}^3 \lambda_\tau ((x + ne_j - (x' - ne_i))^\tau - 1).$$

Recall the definition of the sets  $A, A_1, A_2$  and  $A_3$ . For  $k$  with  $\{i, j, k\} = \{1, 2, 3\}$  we obtain  $L^g(\gamma_n|_{A_k}) \geq \lambda_k(x + ne_j - (x' - ne_i))^k - 1 - 3\varepsilon$ . If this was not true, we would obtain with the bounds  $L^g(\gamma_n|_A) \leq 2\varepsilon$ ,  $L^g(\gamma_n|_{A_i}) \leq \lambda_i(x + ne_j - (x' - ne_i))^i$  and  $L^g(\gamma_n|_{A_j}) \leq \lambda_j(x + ne_j - (x' - ne_i))^j$  that

$$\begin{aligned} \sum_{\tau} \lambda_\tau ((x + ne_j - (x' - ne_i))^\tau - 1) \\ \leq L^g(\gamma_n) \leq \sum_{\tau} \lambda_\tau (x + ne_j - (x' - ne_i))^\tau + 2\varepsilon - 1 - 3\varepsilon. \end{aligned}$$

This is obviously a contradiction.

For  $\delta \in (0, \varepsilon]$  set  $A_{j,\delta} := \gamma_n^{-1}(B_\delta(l'))$ . Recall the definition of  $\eta(\delta)$  from the proof of lemma 8.11. We have

$$L^g(\gamma_n|_{A_j}) \leq (\lambda_j - \eta(\delta)) \int_{A_j \setminus A_{j,\delta}} \dot{\gamma}_n^j + \lambda_j \int_{A_{j,\delta}} \dot{\gamma}_n^j.$$

From proposition 8.6 we have  $L^g(\gamma_n|_{A_j}) \geq \lambda_j \int_{A_j} \dot{\gamma}_n^j - 1 - 8\varepsilon$  and consequently

$$(15) \quad \int_{A_j \setminus A_{j,\delta}} \dot{\gamma}_n^j \leq \frac{1 + 8\varepsilon}{\eta(\delta)}.$$

With lemma 8.11 we see that any limit curve  $\gamma$  of the  $\gamma_n$ 's is asymptotic to  $l'$  for  $t \rightarrow \infty$ . The same argument applies to  $l$  for  $t \rightarrow -\infty$ . Note that the  $\bar{g}$ -length of  $\gamma$  is not bounded. This proves the proposition.  $\square$

**Proposition 8.12.** *For each pair of lines  $l, l' \subseteq L_i$  ( $i \in \{1, 2, 3\}$ ) with  $x^j(l') > x^j(l)$  and  $x^k(l') > x^k(l)$  ( $\{i, j, k\} = \{1, 2, 3\}$ ) there exists a maximal future pointing geodesic  $\gamma: \mathbb{R} \rightarrow \mathbb{R}^3$  asymptotic to  $l$  for  $t \rightarrow -\infty$  and asymptotic to  $l'$  for  $t \rightarrow \infty$ .*

**Proposition 8.13.** *Let  $\zeta$  be a future pointing maximizer asymptotic to a periodic maximizer  $\xi$ . Then  $\zeta$  cannot cross any other periodic maximizer  $\chi$  of the same fundamental class as  $\xi$ .*



*Proof.* The original proof for Riemannian manifolds of dimension two is due to Morse in [15]. The arguments therein work literally in the same way for this case, taking into account that the lines in  $L$  are the traces of lifted periodic timelike maximizers.  $\square$

*Proof of proposition 8.12.* Obviously we have  $l' \in J^+(l)$ . Choose a  $k \in \mathbb{Z}^3$  such that  $l + k = l'$  and a point  $p \in l$ . Further choose maximal future pointing pregeodesics  $\gamma_n: [0, T_n] \rightarrow \mathbb{R}^3$  with  $|\dot{\gamma}_n| \equiv 1$  connecting  $p - ne_i$  to  $p + k + ne_i$ . Let  $[0, a_n]$  and  $(b_n, T_n]$  be maximal intervals with  $(\varepsilon' \in (0, \varepsilon])$  as in lemma 8.11)

$$\gamma_n([0, a_n]) \subseteq B_{\varepsilon'}(l) \text{ and } \gamma_n((b_n, T_n]) \subseteq B_{\varepsilon'}(l').$$

We know with lemma 8.11 that  $\gamma_n$  does not intersect  $B_{\varepsilon'}(l \cup l')$  on  $[a_n + r(\varepsilon'), b_n - r(\varepsilon')]$ .  $\gamma_n$  cannot intersect the  $\varepsilon$ -tube of any other line  $l'' \in L_i$  besides  $l$  and  $l'$  by proposition 8.6. The Lebesgue measure of  $\gamma^{-1}(B_{\varepsilon}(l \cup l') \setminus B_{\varepsilon'}(l \cup l'))$  is bounded with (15). Therefore  $b_n - a_n$  will be bounded, say by  $A > 0$  for all  $n \in \mathbb{N}$ . Next choose integers  $k_n \in \mathbb{Z}$  such that  $\gamma_n(a_n) + k_n e_i$  is bounded in  $\mathbb{R}^3$ . Then we can choose, up to a subsequence, a pregeodesic  $\gamma$  with  $\lim \dot{\gamma}_n(a_n) = \dot{\gamma}(0)$ . If the sequences  $\{a_n\}$  and  $\{T_n - b_n\}$  diverge to infinity, the proof is complete. In more detail: In this case  $\gamma$  will be maximal and  $\gamma(t)$  will be contained in  $\overline{B_{\varepsilon'}(l)}$  for  $t \leq 0$  and in  $\overline{B_{\varepsilon'}(l')}$  for  $t \geq A$ . Lemma 8.11 then shows that  $\gamma$  is asymptotic to  $l$  for  $t \rightarrow -\infty$  and to  $l'$  for  $t \rightarrow \infty$ .

To prove the proposition we have to exclude the other cases (a)  $\{a_n\}$  is bounded and (b)  $\{T_n - b_n\}$  is bounded. This works completely analogously to the proof of proposition 5.7 in [2], using proposition 8.13. Again the unboundedness of the  $g$ -length of  $\gamma$  implies the proposition.  $\square$

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